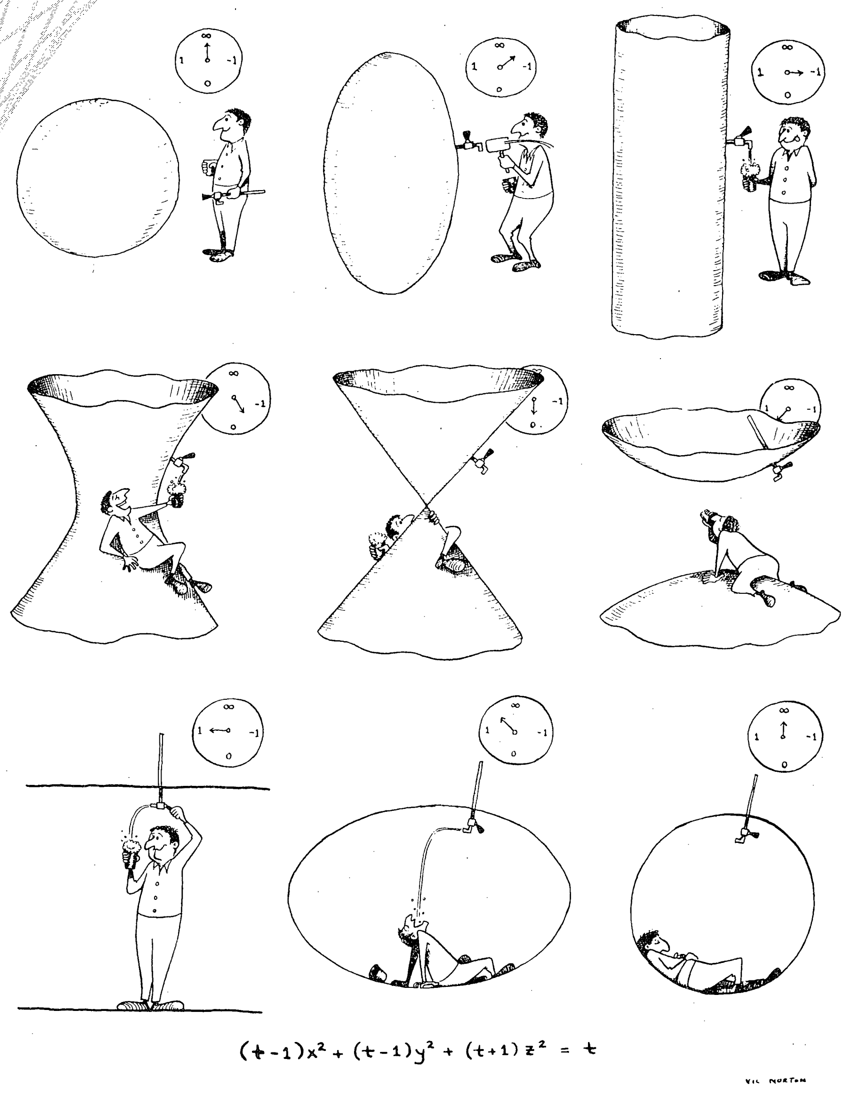
MATHEMATICS MAGAZINE



Vol. 51, No. 4
September, 1978
CODEN: MAMGA8

SQUARE-FREE NUMBERS • SPACE DIVIDED

RENCONTRE • PROBABLE PROBABILITIES

MATHEMATICS MAGAZINE

Volume 51, Number 4, September 1978

COVER: This drawing by Vic Norton of Bowling Green State University illustrates a little known but serendipitous application of eccentricity.

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EDITORIAL POLICY

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Authors planning to submit manuscripts may find it helpful to obtain the more detailed statement of guidelines available from the editorial office.

ABOUT OUR AUTHORS

[**Karl Greger**] ("Square Divisors and Square-free Numbers") graduated from the University of Lund in Sweden. He has been responsible for the training of college teachers in mathematics at the University of Göteborg, Sweden, since 1962. This article is part of his endeavour to make the teaching of mathematics more problem-oriented and thereby more stimulating for prospective teachers and their future pupils. His main interests are the theory of numbers, the theory of probability and applied mathematics.

[**Gerald L. Alexanderson and John E. Wetzel**] ("Simple Partitions of Space"), though trained in other areas at Stanford, became interested in combinatorial problems in geometry through contact with George Pólya. Separately and jointly, they have published a number of papers in recent years on partitions of geometrical figures, including a joint paper entitled "Dissections of a Plane Oval" that appeared in June-July, 1977, in the *American Mathematical Monthly*. The authors share with Professor Pólya an interest in heuristics and problem-solving, and much of their work has a problem-solving flavor. Professor Alexanderson is Chairman of the Department of Mathematics at the University of Santa Clara; Professor Wetzel is Associate Professor of Mathematics at the University of Illinois in Urbana-Champaign.

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ARTICLES

Square Divisors and Square-free Numbers

Computer experiments suggest conjectures which may be confirmed probabilistically and then proved by exact analytic means.

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This paper deals with a topic from analytic number theory: the square divisors of an integer and the distribution of the square-free numbers. Our aim is to show that problems in this field can be made accessible in college by a combination of computer experiments and probabilistic reasoning, enabling students to gain insight into the problems and to "see" and "guess" theorems before embarking upon a more systematic study of the subject.

1. Some problems and computer experiments

Every positive integer n permits a unique factorization of the form

$$n = D^2 \cdot U \tag{1}$$

where D^2 is the largest square divisor of n. D is called the **square component** of n:

$$D = D(n) = \max_{d^2|n} d, \tag{2}$$

and

$$U = U(n) = n/D^2 \tag{3}$$

is the **square-free component** of n. An integer n is said to be **square-free** if its square component $D^2(n) = 1$, i.e., if $n = U(n)$.

The square-free integers are easily obtained by a simple sieving process, the ***Q*-algorithm**, which is a combination of two operations: a marking operation by which square-free integers are marked ("encircled") and a cancelling operation by which numbers with a square divisor greater than 1 are cancelled. To obtain all square-free integers less than or equal to n, the algorithm takes the following form:

The Q-algorithm

[Q1] Mark 1

[Q2] $k \leftarrow$ the least non-cancelled and unmarked number

[Q3] if $k^2 > n$, then [Q7]

[Q4] Mark k

[Q5] Cancel all multiples of k^2

[Q6] Go to [Q2]

[Q7] End

After application of this algorithm to the set of positive integers, the set of non-cancelled numbers less than or equal to n consists of exactly the square-free integers not exceeding n. (The square-free integers less than or equal to $\sqrt{n}$ are also marked.)

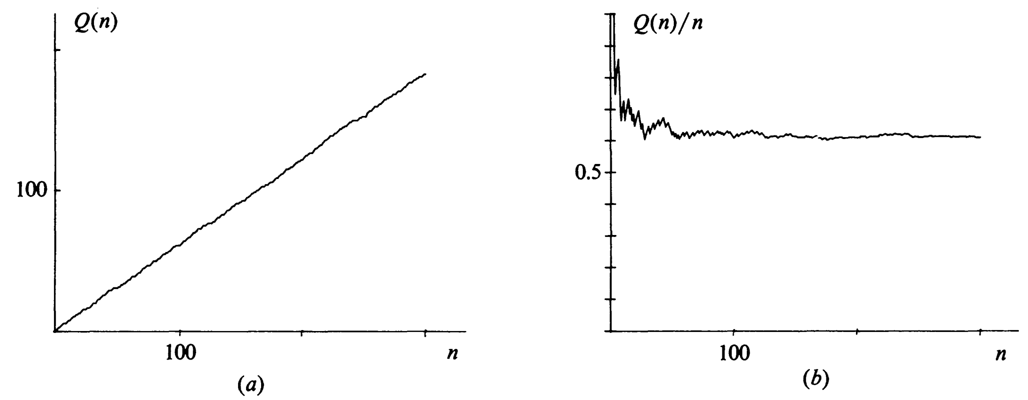

FIGURE 1

Simple experiments reveal that while the distribution of the square-free integers is highly irregular, their density appears to be constant. Let $Q(n)$ denote the number of square-free integers less than or equal to n. FIGURE 1 shows a computer's plotting of $Q(n)$ and $Q(n)/n$ for $1 \leqslant n \leqslant 300$. These empirical results lead to a first conjecture:

CONJECTURE 1. $Q(n) \approx C \cdot n$.

In Section 2 we will show that actually $Q(n) = Cn + O(\sqrt{n}\,)$ and determine the constant C, both by heuristic probabilistic arguments and in a more rigorous way.

Another computer experiment, which can lead students to surprising discoveries and non-trivial theoretical activities, is an empirical investigation of the square-free component $U(n)$ of n. FIGURE 2a shows the result of such an investigation for $1 \leqslant n \leqslant 300$. It is very difficult to see any pattern in FIGURE 2a, so this is the right moment to introduce a statistical idea—the smoothing of observations—by suggesting an investigation of the average square-free component $U_1(n)$, defined by

$$U_1(n) = \frac{1}{n} \cdot \sum_{k=1}^{n} U(k). \tag{4}$$

The result is startling (FIGURE 2b) and immediately leads to a new conjecture:

CONJECTURE 2. $U_1(n) \approx C \cdot n$.

The truth of this conjecture will be revealed in Section 3.

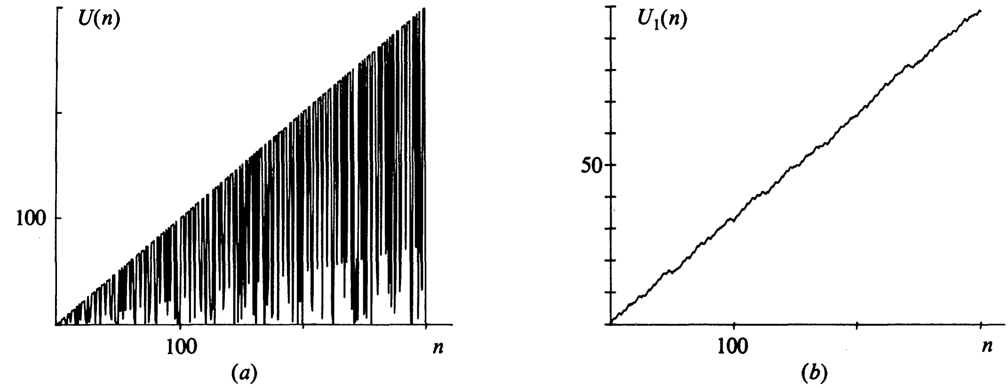

FIGURE 2

In a similar way one can explore the behaviour of the square component $D^2(n)$, or of its square root $D(n)$, and their averages, but here it is somewhat more difficult to make conjectures. This problem will be treated in Section 5. A quite unexpected discovery can be made by investigating the number $A(n)$ of square divisors of an integer n. FIGURE 3a shows $A(n)$ for $1 \leqslant n \leqslant 300$, while FIGURE 3$b$ shows the average number of square divisors $A_1(n)$, defined by

$$A_1(n) = \frac{1}{n} \cdot \sum_{k=1}^{n} A(k). \tag{5}$$

This result suggests a new conjecture to be dealt with in Section 4:

CONJECTURE 3. $A_1(n) \approx C$.

2. The density of the square-free numbers

Let $Q(n)$ be the number of square-free integers less than or equal to n. Their **density** $q(n)$ is defined by

$$q(n) = Q(n)/n. \tag{6}$$

We want to investigate the asymptotic behaviour of $q(n)$ as $n \to \infty$. Conjecture 1 implies that $q(n)$ converges to a limit C as $n \to \infty$. This conjecture will be verified below, with the limit C computed both by heuristic probabilistic considerations and in a more rigorous way.

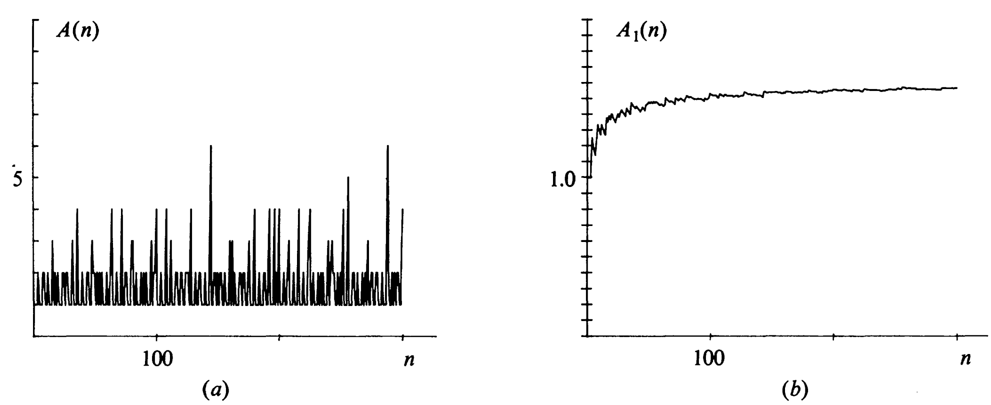

FIGURE 3

In what follows we often use the "big oh" notation which, for our purpose, can be explained thus: if $F(n)$ is an arithmetical function, then $F(n)=O(n^c)$ (read "$F(n)$ is big oh of n^c") shall mean that there is a constant C, independent of n, such that $|F(n)| \leq C \cdot n^c$ for all sufficiently large n.

Heuristics. The density $q(n)$ may be interpreted as the probability that a randomly chosen integer less than or equal to n is square-free. Let d be a randomly chosen integer less than or equal to $\sqrt{n}$ and consider the event E that d^2 is the largest square divisor of n. This event will take place only if simultaneously

a) d^2 is a divisor of n, which occurs with probability $1/d^2$, and
b) n/d^2 is square-free, which occurs with probability $q(n)$.

Thus the event E occurs with probability $q(n)/d^2$.

The fact that some integer $d^2 \leq n$ must be the largest square divisor of n implies that

$$\sum_{d=1}^{[\sqrt{n}\,]} q(n)/d^2 = 1. \tag{7}$$

It is a well-known fact that $\sum_{k=1}^{n} 1/k^2 = \pi^2/6 + O(1/n)$ (cf [5] for a short elementary proof) which together with (7) yields $q(n)\cdot(\pi^2/6 + O(1/\sqrt{n}\,)) = 1$ and consequently

$$q(n) = 6/\pi^2 + O(1/\sqrt{n}\,) = 0.6079\cdots + O(1/\sqrt{n}\,). \tag{8}$$

Our result is in excellent agreement with observations of the ratio of the square-free integers in a series of randomly chosen integers between 1 and 1000000 (see TABLE 1).

Number of randomly chosen integers between 1 and 10^6	Number of square-free integers	Ratio of square-free integers
50	31	0.62
100	63	0.63
150	89	0.593

TABLE 1

Proof of Conjecture 1. In order to be square-free, an integer n must not be divisible by the square of any prime. The principle of inclusion and exclusion (cf. chapter 2 in [7]) yields immediately the following equality for $Q(n)$, the number of square-free integers not exceeding n:

$$Q(n) = n - \sum \left[\frac{n}{p^2} \right] + \sum \left[\frac{n}{p_1^2 p_2^2} \right] - \sum \left[\frac{n}{p_1^2 p_2^2 p_3^2} \right] + \cdots. \tag{9}$$

In these sums $p, p_1, p_2, \ldots$ run through all primes not exceeding $\sqrt{n}$, but in each denominator $p_1, p_2, p_3, \ldots$ are supposed to be different primes satisfying $p_1 < p_2 < p_3 < \cdots$.

Let $\nu(x)$ denote the number of different prime divisors of the integer x. The Möbius function μ is defined for all positive integers n by

$$\mu(n) = \begin{cases} (-1)^{\nu(n)} \text{ if } n \text{ is square-free,} \\ 0 \text{ otherwise.} \end{cases}$$

Introducing μ in (9), $Q(n)$ can be written

$$Q(n) = \sum_{k=1}^{[\sqrt{n}\,]} \mu(k) \cdot \left[\frac{n}{k^2} \right].$$

The number of terms in this sum being less than or equal to $\sqrt{n}$, the error committed by dropping the integral-part operator [ ] is of order $O(\sqrt{n}\,)$. Therefore

$$Q(n)=\sum_{k=1}^{[\sqrt{n}]}\mu(k)\cdot n/k^2+O(\sqrt{n}). \tag{10}$$

By introducing the density $q(n)=Q(n)/n$, equality (10) takes the form

$$\begin{aligned}q(n)&=\sum_{k=1}^{[\sqrt{n}]}\mu(k)/k^2+O(1/\sqrt{n})\\&=\sum_{k=1}^{\infty}\mu(k)/k^2+R+O(1/\sqrt{n}),\end{aligned}$$

where $|R|\leqslant\sum_{[\sqrt{n}]}^{\infty}1/k^2=O(1/\sqrt{n})$. Therefore

$$q(n)=\sum_{k=1}^{\infty}\mu(k)/k^2+O(1/\sqrt{n}).$$

Taking into account that

$$\sum_{k=1}^{\infty}\mu(k)/k^2=\prod_{p\text{ prime}}(1-p^{-2}),$$

$q(n)$ can be written

$$q(n)=\prod_{p\text{ prime}}(1-p^{-2})+O(1/\sqrt{n}). \tag{11}$$

Rewriting and inverting (11) yields

$$\begin{aligned}\frac{1}{q(n)-O(1/\sqrt{n})}&=\prod_{p\text{ prime}}\frac{1}{1-p^{-2}}\\&=\prod_{p\text{ prime}}(1+p^{-2}+p^{-4}+\cdots)\\&=\sum_{k=1}^{\infty}1/k^2=\pi^2/6.\end{aligned} \tag{12}$$

Inverting (12) concludes our proof that

$$q(n)=\frac{6}{\pi^2}+O(1/\sqrt{n})\quad\text{and}\quad Q(n)=\frac{6}{\pi^2}\cdot n+O(\sqrt{n}). \tag{13}$$

(This theorem is due to Gegenbauer **[2]**.)

3. The square-free component of an integer

If D^2 is the largest square divisor of an integer n, then $U(n)=n/D^2$ is called, by the definition given in Section 1, the square-free component of n. Computer experiments described in that section lead to the conjecture that the average square-free component $U_1(n)$, defined by (4), satisfies $U_1(n)\sim C\cdot n$. We shall now verify this conjecture and compute the constant C.

Heuristics. Let d be a randomly chosen integer less than or equal to $\sqrt{n}$. The event E of d^2 being the largest square divisor of n (and, incidentally, of n/d^2 being the square-free component of n) will take place only if

a) d^2 is a divisor of n, which occurs with probability $1/d^2$, and
b) n/d^2 is a square-free number, which according to the results of Section 2 will occur with probability $q(n)=6/\pi^2+O(1/\sqrt{n})$.

Therefore n/d^2 is the square-free component of n with probability $q(n)/d^2$.

We can now introduce the "expected" square-free component $u(n)$ of n by

$$u(n)=\sum_{d\leq\sqrt{n}}\frac{n}{d^2}\cdot\frac{q(n)}{d^2}=n\cdot q(n)\cdot\sum_{d=1}^{[\sqrt{n}]}d^{-4}$$

$$=n\cdot q(n)\cdot\left(\frac{\pi^4}{90}+O(n^{-3/2})\right)=Q(n)\cdot\frac{\pi^4}{90}+Q(n)\cdot O(n^{-3/2})$$

$$=\frac{\pi^2}{15}\cdot n+O(\sqrt{n}\,). \tag{14}$$

(Here we have made use of the fact that $\sum_{k\leq x}k^{-4}=\pi^4/90+O(x^{-3})$.) If in (4) we allow ourselves to substitute the expected square-free component $u(n)$ for the actual square-free component $U(n)$, we obtain

$$U_1(n)=\frac{1}{n}\cdot\sum_{k=1}^{n}u(k)=\frac{\pi^2}{30}\cdot n+O(\sqrt{n}\,)=(0.3289\ldots)n+O(\sqrt{n}\,), \tag{15}$$

in excellent agreement with the observations of TABLE 2. (15) is a special case of a theorem in Cohen [1].

The probabilistic arguments in this section can be tightened up considerably by introducing the concepts of random divisor and conjugate random divisor. For details, in another context but immediately carried over here, the reader is referred to [3].

n	$U_1(n)$	$U_1(n)/n$
100	32.33	0.3233
200	65.435	0.3272
300	98.42	0.3281

TABLE 2

Proof of Conjecture 2. As before, let $Q(n)$ be the number of square-free integers not exceeding n, define $Q(0)=0$ and introduce the **square-free indicator function** g by

$$g(n)=Q(n)-Q(n-1). \tag{16}$$

Then $g(n)=1$ if n is square-free and $g(n)=0$ otherwise. For each integer k less than or equal to $\sqrt{n}$, the sum s_k of the square-free components of those integers less than or equal to n having k^2 as their largest square divisor is

$$s_k=\sum_{j=1}^{[n/k^2]}j\cdot g(j)=\left[\frac{n}{k^2}\right]\cdot Q\left(\left[\frac{n}{k^2}\right]\right)-\sum_{j=1}^{[n/k^2]-1}Q(j)$$

$$=\frac{3}{\pi^2}\cdot\left[\frac{n}{k^2}\right]^2+O(n^{3/2}k^{-3})$$

by Abel's method of partial summation and (13). This yields the following expression for the average square-free component $U_1(n)$:

$$U_1(n)=\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}]}s_k=\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}]}\sum_{j=1}^{[n/k^2]}j\cdot g(j)$$

$$=\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}]}\frac{3}{\pi^2}\left[\frac{n}{k^2}\right]^2+\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}]}O(n^{3/2}k^{-3}). \tag{17}$$

The number of terms in each of the two last sums being less than or equal to $\sqrt{n}$, it is easily verified that the error committed by dropping the integral-part operator [ ] is of order $O(\sqrt{n}\,)$. Therefore

$$U_1(n)=\frac{3}{n\pi^2}\cdot\sum_{k=1}^{[\sqrt{n}]}n^2k^{-4}+O(\sqrt{n}\,)+O(\sqrt{n}\,). \tag{18}$$

The fact that $\Sigma_{k=1}^{N} k^{-4} = \frac{\pi^4}{90} + O(N^{-3})$ yields, finally,

$$U_1(n) = \frac{\pi^2}{30} \cdot n + O(\sqrt{n}), \tag{19}$$

which verifies Conjecture 2.

4. The number of square divisors of an integer

Let $A(n)$ be the number of square divisors of the integer n. Computer experiments, described in Section 1, led to Conjecture 3 that the average number of square divisors $A_1(n)$, defined by (5), converges to a limit as $n \to \infty$. In this section we shall verify this conjecture and compute $\lim_{n\to\infty} A_1(n)$.

Heuristics. Let d be a randomly chosen positive integer less than or equal to $\sqrt{n}$. Its square d^2 is a divisor of n with probability $1/d^2$. Let $a(n)$ be the "expected" number of square divisors of n. Because every positive integer d not exceeding $\sqrt{n}$ contributes 1 with probability $1/d^2$ to the number of square divisors of n,

$$a(n) = \sum_{d=1}^{[\sqrt{n}]} 1 \cdot d^{-2} = \frac{\pi^2}{6} + O(1/\sqrt{n}). \tag{20}$$

Substituting $a(n)$ for $A(n)$ in (5) makes it plausible that

$$A_1(n) = \frac{\pi^2}{6} + O(1/\sqrt{n}) = 1.6449\cdots + O(1/\sqrt{n})$$

in good agreement with empirical observations of the average number of square divisors of randomly chosen integers reproduced in TABLE 3.

Number of integers randomly chosen between 1 and 1000000	Average number of square divisors of observed integers
50	1.54
100	1.61
150	1.64

TABLE 3

Proof of Conjecture 3. Let d be any positive integer less than or equal to $\sqrt{n}$. The number of integers not exceeding n divisible by d^2 is $[n/d^2]$. Therefore, the average number of square divisors of n is

$$A_1(n) = \frac{1}{n} \cdot \sum_{d=1}^{[\sqrt{n}]} [n/d^2]. \tag{21}$$

The number of terms on the right hand side of (21) being less than or equal to $\sqrt{n}$, the error introduced by dropping the integral-part operator in (21) is of order $O(1/\sqrt{n})$:

$$A_1(n) = \frac{1}{n} \cdot \sum_{d=1}^{[\sqrt{n}]} n/d^2 + O(1/\sqrt{n})$$

$$= \sum_{d=1}^{[\sqrt{n}]} 1/d^2 + O(1/\sqrt{n}).$$

Since $\Sigma_{k=1}^{n} 1/k^2 = \pi^2/6 + O(1/n)$, as we noted above immediately after equation (7), one can write

$$\sum_{d=1}^{[\sqrt{n}]} 1/d^2 = \pi^2/6 + O(1/\sqrt{n}).$$

Therefore $A_1(n)$ can be written as

$$A_1(n)=\pi^2/6+O(1/\sqrt{n}\,),$$

which verifies Conjecture 3.

5. The square component of an integer

As in equation (2), let $D(n)=\max_{d^2|n} d$ and define $D_1(n)$ by

$$D_1(n)=\frac{1}{n}\cdot\sum_{k=1}^{n} D(k). \tag{22}$$

Our aim is to find an asymptotic expression for $D_1(n)$. As usual, we begin with some heuristic probabilistic considerations. In order to find the "expected" value $t(n)$ of $D(n)$, the positive integer k less than or equal to $\sqrt{n}$ is chosen at random. Then k^2 will be the largest square divisor of n only if

a) k^2 is a divisor of n, which occurs with probability $1/k^2$, and
b) n/k^2 is square-free, which occurs with probability $q(n/k^2)=6/\pi^2+O(k/\sqrt{n}\,)$.

It follows that k^2 is the largest square divisor of n with probability $q(n/k^2)/k^2$ and, therefore, the expected value $t(n)$ of $D(n)$ will be

$$\begin{aligned} t(n)&=\sum_{k=1}^{[\sqrt{n}\,]} k\cdot q(n/k^2)/k^2=\sum_{k=1}^{[\sqrt{n}\,]}(6/\pi^2)/k+O(1)\\ &=\frac{3}{\pi^2}\cdot\log n+O(1). \end{aligned} \tag{23}$$

Substitution of $t(n)$ for $D(n)$ in (22) makes it plausible that

$$D_1(n)=\frac{1}{n}\cdot\sum_{k=1}^{n} t(k)=\frac{3}{\pi^2}(\log n+O(1)).$$

To derive this result in a more rigorous way, one observes that the number of those integers not exceeding n having k^2 as their largest square divisor is $Q([n/k^2])$. Consequently

$$\begin{aligned} D_1(n)&=\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}\,]} k\cdot Q([n/k^2])\\ &=\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}\,]} k\cdot\left(\frac{6}{\pi^2}\left[\frac{n}{k^2}\right]+O(\sqrt{n}\,/k)\right)\\ &=\frac{6}{\pi^2 n}\cdot\sum_{k=1}^{[\sqrt{n}\,]} k\cdot\left[\frac{n}{k^2}\right]+\frac{1}{n}\cdot\sum_{k=1}^{[\sqrt{n}\,]} O(\sqrt{n}\,). \end{aligned}$$

Dropping the integral-part operator incurs an error of order $O(1)$ and thus

$$\begin{aligned} D_1(n)&=\frac{6}{\pi^2 n}\cdot\sum_{k=1}^{[\sqrt{n}\,]}\frac{n}{k}+O(1)+O(1)\\ &=\frac{6}{\pi^2}\cdot\log\sqrt{n}\,+O(1)=\frac{3}{\pi^2}\cdot\log n+O(1). \end{aligned} \tag{24}$$

Using more refined methods in [8], Sierpinski derives:

$$D_1(n)=\frac{3}{\pi^2}\cdot\log n+\frac{9\gamma}{\pi^2}+\frac{36}{\pi^4}\cdot\sum_{k=1}^{\infty}\frac{\log k}{k^2}+o(1),$$

where $\gamma=0{,}577\ldots$ is Euler's constant and $o(1)$ represents a function that goes to zero as $n\to\infty$.

6. A special sum

Define $S(n)$ by

$$S(n)=\sum_{d^2|n} d \tag{25}$$

and the corresponding average $S_1(n)$ by

$$S_1(n)=\frac{1}{n}\cdot\sum_{k=1}^{n} S(k). \tag{26}$$

To find the "expected" value $s(n)$ of $S(n)$, let the positive integer d less than or equal to $\sqrt{n}$ be chosen at random. Then d^2 divides n with probability $1/d^2$, and therefore

$$s(n)=\sum_{d=1}^{[\sqrt{n}\,]} d\cdot(1/d^2)=\sum_{d=1}^{[\sqrt{n}\,]} 1/d=\frac{1}{2}\log n+O(1).$$

Substituting $s(k)$ for $S(k)$ in (26) makes it plausible that

$$S_1(n)=\frac{1}{2}\log n+O(1).$$

To prove this conjecture, one has to observe that there are $[n/d^2]$ integers not exceeding n divisible by d^2. Therefore

$$S_1(n)=\frac{1}{n}\cdot\sum_{d=1}^{[\sqrt{n}\,]} d\cdot[n/d^2]=\frac{1}{n}\cdot\sum_{d=1}^{[\sqrt{n}\,]}\frac{n}{d}+R$$

with $|R|\leqslant\frac{1}{n}\cdot\sum_{d=1}^{[\sqrt{n}\,]} d=O(1)$. Hence, finally,

$$S_1(n)=\frac{1}{2}\cdot\log n+O(1). \tag{27}$$

Using more refined methods, Sierpinski proves in [8] that

$$S_1(n)=\frac{1}{2}\log n+\frac{3\gamma}{2}+o(1),$$

γ being Euler's constant. Additional examples of arithmetic functions with relationships discoverable by numerical experiments (and open to heuristic justification) may be found in [3].

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Simple Partitions of Space

As a general plane sweeps through space it counts the various types of regions produced by a given collection of planes.

This article is respectfully dedicated to George Polya on the occasion of his 90th day, December 13, 1977.

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Consider an arrangement of n planes in general position in Euclidean 3-space, where by "general position" we mean that at most three of the planes pass through each point, no more than two of the planes pass through each line, there are no parallel planes, and no lines of intersection are parallel to planes of the arrangement. Into how many cells is space divided?

This problem is studied from the heuristic point of view by G. Pólya in [4] and again in his film, "Let Us Teach Guessing." The solution,

$$C=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3},$$

was found in 1826 by J. Steiner [6]. We shall give a simple proof of a more general result using an ingenious technique developed by A. Brousseau [1] but introduced earlier for a somewhat different purpose by H. Hadwiger [3].

An arrangement of n planes in space is **simple** provided no more than three planes pass through any point of intersection and no more than two planes pass through any line of intersection. A completely arbitrary arrangement of planes in space can have various kinds of degeneracies: there could be "multiple" points (i.e., points through which more than three planes pass); there could be "multiple" lines (lines through which more than two planes pass); there could be parallel planes; or there could be planes parallel to the line of intersection of other planes. The first two kinds of degeneracies are ruled out in a simple arrangement, but the second two are permitted.

The planes of a simple arrangement intersect to form intersections of various dimensions. Let C be the number of three-dimensional cells formed by the planes, F the number of faces (two-dimensional regions) formed by the lines of intersection on the planes, E the number of edges (segments and rays) formed on the lines of intersection by the points in which the planes meet, and V the number of points (vertices) determined by the planes. Our first objective is to find formulas for C, F, and E in terms of appropriate data.

The Plane

In studying a problem in 3-space, one can frequently gain insight from a thoughtful examination of its analogue in, or specialization to, the plane. So we first consider the problem of finding the number R of regions and the number S of segments and rays formed in the plane by a simple arrangement of lines.

An arrangement of lines in the plane is **simple** provided no more than two lines pass through any point. A completely arbitrary arrangement of lines in the plane can have multiple points (i.e., points through which more than two lines pass) or parallel lines. The first kind of degeneracy is ruled out in a simple arrangement, but parallels are permitted.

THEOREM 1. *If the n lines in a simple arrangement meet to form p points of intersection, then they partition the plane into,*

$$R=1+n+p \quad \textit{regions}$$
$$S=n+2p \quad \textit{segments and rays, and}$$
$$P=p \quad \textit{points}$$

Proof. Consider an auxiliary line, chosen so as not to be parallel to any of the n lines of the arrangement. With Brousseau, we call this line a sweep-line. If initially this sweep-line is positioned well outside the bounded regions, it is intersected by each of the n lines and cut by the points of intersection into $1+n$ segments and rays, each of which lies in (and so counts) a region in the plane. As the sweep-line is moved parallel to its initial position across the plane, it enters a new region of the arrangement precisely when it passes through a point of intersection, and at each such point it picks up exactly one new region because the arrangement is simple. Since there are p points, there must be equally many new regions. So $R=1+n+p$.

One can apply this same technique to count the segments and rays on the partitioning lines. The sweep-line initially (that is, far removed from the bounded regions) accounts for n rays, one on each of the partitioning lines. As the sweep-line passes across the plane, it picks up two new segments or rays at each of the p points of intersection. Therefore, $S=n+2p$.

Space

Now we try the analogous argument with a sweep-plane in space.

THEOREM 2. *If the n planes in a simple arrangement in space meet to form p points of intersection and l lines of intersection, then they partition space into*

$$C=1+n+l+p \quad \textit{cells,}$$
$$F=n+2l+3p \quad \textit{faces,}$$
$$E=l+3p \quad \textit{edges, and}$$
$$V=p \quad \textit{points.}$$

Proof. We imitate the earlier argument and introduce a sweep-plane, which we pick not parallel to any of the partitioning planes or to any of their lines of intersection and located initially far removed from the bounded cells. In its initial position, this plane counts certain cells, faces, and edges. Each region on the sweep-plane corresponds to one cell (which it cuts in two), each segment or ray on the sweep-plane corresponds to a face of the arrangement, and each point on the sweep-plane corresponds to a ray on the arrangement. Therefore, by Theorem 1, the sweep-plane initially counts $1+n+l$ cells, $n+2l$ faces, and l edges. To justify this application of Theorem 1, one has to observe that the lines produced in the sweep-plane by the given planes form a simple arrangement, but this is easy to see.

Now, as the sweep-plane passes through space moving parallel to its initial position, it picks up one new cell, 3 new faces (one on each plane passing through the point), and 3 new edges (one on each line of intersection through the point) at each point of intersection, because the arrangement is simple. Since there are p such points, the sums increment by exactly p cells, $3p$ faces, and $3p$ edges. This completes the argument.

The generalization of this result to higher dimensions is strongly suggested by the patterns. The analogue in Euclidean d-dimensional space E^d of a point on a line, a line in the plane, and a plane in space is called a "hyperplane." It has dimension $d-1$, and it has the important property that it splits E^d into two parts, called halfspaces. The objects in E^d analogous to

points	in	E^1
points and lines	in	E^2
points, lines, and planes	in	E^3,

are called "k-flats," where k indicates the dimension; so in E^d, a 0-flat is a point, a 1-flat is a line, a 2-flat is a plane,..., a $(d-1)$-flat is a hyperplane.

An arrangement of n hyperplanes in E^d is **simple** if there are no multiple k-flats for any k (precisely, we demand for each k that no more than $d-k$ of the hyperplanes be "concurrent" in any k-flat). Parallelisms of various sorts are all permitted; we do not demand that each $d-k$ hyperplane actually intersect to form a k-flat. Then these n hyperplanes form a partition of E^d that has, for each r with $0 \leqslant r \leqslant d$, exactly

$$C_r^d(n) = \sum_{k=d-r}^{d} \binom{k}{d-r} f_k$$

r-dimensional cells, where $f_d = 1, f_{d-1} = n$, and for each k with $0 \leqslant k \leqslant d-2, f_k$ is the number of k-flats formed by the arrangement of hyperplanes. A proof using some sophisticated algebraic machinery may be found in Zaslavsky [7], but a convincing intuitive argument can be based on the sweep-hyperplane method.

Steiner Data

The n lines (or planes) in a simple arrangement fall naturally into parallel families (the equivalence classes of the parallelism relation, if we agree to say that a line (plane) is parallel to itself). If there are s such parallel families having $n_1, n_2, \ldots, n_s$ lines (planes) respectively (remember, we allow $n_i = 1$) we call the s-tuple $\langle n_1, n_2, \ldots, n_s \rangle$ the **Steiner data** for the simple arrangement.

Now suppose a simple arrangement of n lines in the plane has Steiner data $\langle n_1, n_2, \ldots, n_s \rangle$. It is easy to see that these

$$\sigma_1 = \sigma_1(n_1, n_2, \ldots, n_s) = \sum_{1 \leqslant i \leqslant s} n_i = n$$

lines meet to determine

$$\sigma_2 = \sigma_2(n_1, n_2, \ldots, n_s) = \sum_{1 \leqslant i < j \leqslant s} n_i n_j$$

points of intersection.

THEOREM 3. *A simple arrangement of lines in the plane with Steiner data* $\langle n_1, n_2, \ldots, n_s \rangle$ *forms*

$R = \sigma_0 + \sigma_1 + \sigma_2$ *regions,*

$S = \sigma_1 + 2\sigma_2$ *segments and edges, and*

$P = \sigma_2$ *points,*

where $\sigma_0 = 1$, σ_1, *and* σ_2 *are the elementary symmetric functions on the Steiner data.*

Similarly, consider a simple arrangement of n planes in space having Steiner data $\langle n_1, n_2, \ldots, n_s \rangle$.A **director** of the arrangement is a set of s planes, one chosen from each parallel family. We call a simple arrangement **non-degenerate** if it has a director whose planes are in general position. In a non-degenerate arrangement, the $\sigma_1 = \sum_{1 \leqslant i \leqslant s} n_i$ planes intersect to form $\sigma_2 = \sum_{1 \leqslant i < j \leqslant s} n_i n_j$ lines of intersection and $\sigma_3 = \sum_{1 \leqslant i < j < k \leqslant s} n_i n_j n_k$ points of intersection.

THEOREM 4. *A non-degenerate simple arrangement of planes in space with Steiner data* $\langle n_1, n_2, \dots, n_s \rangle$ *forms*

$$C=\sigma_0+\sigma_1+\sigma_2+\sigma_3 \quad \textit{cells},$$
$$F=\sigma_1+2\sigma_2+3\sigma_3 \quad \textit{faces},$$
$$E=\sigma_2+3\sigma_3 \quad \textit{edges, and}$$
$$V=\sigma_3 \quad \textit{vertices},$$

where $\sigma_0=1$, σ_1, σ_2, *and* σ_3 *are the elementary symmetric functions on the Steiner data.*

The formulas for R and C in these corollaries were first given by Steiner, who proved them by induction. It is clear from the pattern what happens in higher dimensions. A non-degenerate simple arrangement of n hyperplanes in E^d with Steiner data $\langle n_1, n_2, \dots, n_s \rangle$ forms exactly

$$C_r^d(n)=\sum_{k=d-r}^{d}\binom{k}{d-r}\sigma_k$$

r-dimensional cells, where σ_k is the kth elementary symmetric function on the Steiner data, because $f_k=\sigma_k$ for each k.

Special Cases

Suppose that the n planes of a non-degenerate simple arrangement fall into s parallel families, all of which have the same number m of planes, so that $n_i=m$ for each i. Then

$$\sigma_1=m\binom{s}{1}=n, \qquad \sigma_2=m^2\binom{s}{2}, \qquad \sigma_3=m^3\binom{s}{3},$$

and we have

$$C=\binom{s}{0}+m\binom{s}{1}+m^2\binom{s}{2}+m^3\binom{s}{3}, \qquad E=m^2\binom{s}{2}+3m^3\binom{s}{3},$$
$$F=m\binom{s}{1}+2m^2\binom{s}{2}+3m^3\binom{s}{3}, \qquad V=m^3\binom{s}{3}.$$

The extended face-planes of a cube, for example, form a simple arrangement of $n=6$ planes that fall into $s=3$ parallel families, each of which contains $m=2$ planes. So, as can easily be confirmed visually,

$$C=1+2\cdot3+2^2\cdot3+2^3\cdot1=27, \qquad E=2^2\cdot3+3\cdot2^3\cdot1=36,$$
$$F=2\cdot3+2\cdot2^2\cdot3+3\cdot2^3\cdot1=54, \qquad V=2^3\cdot1=8.$$

If $m=1$, the $n=s$ planes are in general position, and we have

$$C=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}, \qquad E=\binom{n}{2}+3\binom{n}{3},$$
$$F=\binom{n}{1}+2\binom{n}{2}+3\binom{n}{3}, \qquad V=\binom{n}{3}.$$

In the plane the analogous formulas are

$$R=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}, \qquad S=\binom{n}{1}+2\binom{n}{2}, \qquad P=\binom{n}{2},$$

and in E^d we have, for each r with $0 \leqslant r \leqslant d$,

$$C_r^d(n)=\sum_{k=d-r}^{d}\binom{k}{d-r}\binom{n}{k}$$

r-dimensional cells formed by an arrangement of n hyperplanes in general position. This formula was first given in 1943 by R. C. Buck [2]. The case $r=d$ was given earlier by L. Schläfli [5].

It is interesting that by means of the easy identity

$$\binom{a}{b}\binom{b}{c}=\binom{a}{c}\binom{a-c}{a-b},$$

Buck's formula can be written

$$C_r^d(n)=\binom{n}{d-r}\sum_{k=0}^{r}\binom{n-(d-r)}{k},$$

which involves a sum part way across a row in Pascal's triangle. For example, the number of 5-dimensional cells formed by an arrangement of $n=10$ hyperplanes in general position in E^7 is the product of $\binom{10}{2}=45$ and the sum of the first six entries in the ninth row of Pascal's triangle, viz., 9855.

Bounded Cells

The question of how many of the cells are bounded was also considered in 1826 by Steiner. The situation is considerably more transparent in the plane than in space, so we begin there.

Unless the n lines of an arrangement are all parallel, each line is cut by at least one other line to form $2n$ rays; and since each of these $2n$ rays is the clockwise boundary of an unbounded region, there must be $2n$ such regions. One can also count the unbounded regions in the following way with a sweep-line. There are unbounded regions that correspond to each segment and ray on the sweep-line in initial position, and after the sweep-line has moved through all the bounded regions, there are new unbounded regions that correspond to each segment (but not to the rays) on the sweep-line. Consequently, there must be $(n+1)+(n-1)=2n$ unbounded regions. We shall use the analogous sweep-plane argument in space.

Subtracting off the unbounded rays and regions, we have formulas for the number R' of bounded regions and the number S' of segments formed by the arrangement.

THEOREM 5. *In any simple arrangement of $n \geqslant 2$ lines in the plane not all of which are parallel,*

$$R'=1-n+p=\sigma_0-\sigma_1+\sigma_2,$$
$$S'=-n+2p=-\sigma_1+2\sigma_2.$$

Now consider a simple arrangement of n planes in space. If the planes are all parallel to a line in space, then the arrangement has constant cross-section and forms no bounded cells, faces, or edges at all. Otherwise, there are unbounded cells corresponding to each unbounded region on the sweep-plane and, as one can see with a few moments thought, to each bounded region on the sweep-plane in initial position and again after the sweep-plane has moved through the bounded cells. There are, therefore, $2n+2[1-n+l]=2+2l$ unbounded cells. Similarly, there are unbounded faces corresponding to each ray on the sweep-plane and to twice the number of segments on the sweep-plane. So there are $2n+2[-n+2l]$ unbounded faces. And the number of rays formed is twice the number of lines, $2l$. Subtracting these from the total, we find formulas for the numbers C', F', and E' of bounded cells, faces, and edges:

THEOREM 6. *In any simple arrangement of $n \geqslant 3$ planes in space that are not all parallel to a line,*

$$C'=-1+n-l+p,$$
$$F'=n-2l+3p,$$
$$E'=-l+3p.$$

For non-degenerate simple arrrangements, these formulas can also be expressed in terms of symmetric functions on the Steiner data.

These results extend in the obvious manner to give formulas for the number of bounded r-dimensional cells in simple arrangements of hyperplanes in E^d. In particular, an arrangement of n hyperplanes in general position in E^d forms

$$B_r^d(n)=\sum_{k=d-r}^{d}(-1)^{d+k}\binom{k}{d-r}\binom{n}{k}$$

bounded $r-$dimensional cells. This can be written as

$$B_r^d(n)=(-1)^r\binom{n}{d-r}\sum_{k=0}^{r}(-1)^k\binom{n-(d-r)}{k}=\binom{n}{d-r}\binom{n-d+r-1}{r},$$

the last equality being a well-known identity easily proved by induction.

Euler's Formulas

For any simple arrangement of n planes in space, we have found the formulas

$$\begin{aligned}
C&=1+n+l+p & C'&=-1+n-l+p\\
F&=n+2l+3p & F'&=n-2l+3p\\
E&=l+3p & E'&=-l+3p\\
V&=p & V&=p.
\end{aligned}$$

If one takes alternating signs and adds, one finds that

$$C-F+E-V=1 \quad \text{and} \quad C'-F'+E'-V=-1.$$

These formulas are analogous to Euler's polyhedral formula $f-e+v=2$, where f, e, and v count the faces, edges, and vertices of a convex polyhedron.

In the plane, the corresponding formulas are

$$R-S+P=1 \quad \text{and} \quad R'-S'+P=1.$$

The higher dimensional analogues are also correct, with constant $+1$ when all the r-dimensional cells are counted and $(-1)^d$ when only the bounded r-dimensional cells are counted. The key to the proof is the familiar identity

$$\binom{k}{0}-\binom{k}{1}+\binom{k}{2}-\cdots+(-1)^k\binom{k}{k}=0$$

for $k=1,2,3,\ldots$, which arises most easily from setting $x=-1$ in the binomial expansion of $(1+x)^k$.

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Tabulating All Pythagorean Triples

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Diophantine problems are almost as old as number theory itself. The most widely known is probably the one represented by the relation $x^2+y^2=z^2$; trios of x, y, and z satisfying the equation in integers are (for obvious reasons) called **Pythagorean triples**, and if x, y, and z are relatively prime the triple is called **primitive**. This Diophantine equation is one for which the complete solution is known; the most cited solution (see, e.g., [**1**], [**2**]) is represented in the form

$$z=t\cdot(u^2+v^2), \qquad x=t\cdot(u^2-v^2), \qquad y=2t\cdot u\cdot v, \tag{1}$$

where $u>v$, u and v are relative prime, and t assumes all integral values. Other solutions have been published (e.g., [**3**], [**4**], [**5**]), but they contribute hardly anything towards tabulation of Pythagorean triples. Actual tables are rarely seen, no doubt because it is rather clumsy to operate with multiple parameters.

This difficulty can be eliminated by means of the following observation: $n=x\cdot y/(x+y+z)=t\cdot v\cdot(u-v)$ can assume all integer values and nothing else. Of course, each n may represent several Pythagorean triples. (This observation means, incidentally, that the area-perimeter ratio for any triangle with Pythagorean triples as sides is either integral or half-integral, and also that the classical $3:4:5$ triangle has the least area per unit perimeter of any Pythagorean triple triangle.) The problem now is to solve in integers this pair of Diophantine equations:

$$x\cdot y=n\cdot(x+y+z) \quad \text{and} \quad x^2+y^2=z^2.$$

Elimination of z gives $y=2n+2n^2/(x-2n)$. For x and y to be integral, $x-2n$ must be integral and a divisor of $2n^2$; call this divisor d, and then to avoid exchange of x and y consider only such d as satisfies $d<n\sqrt{2}$.

We then have a program for the non-redundant listing of all Pythagorean triples with $x<y<z$: we let $x=2n+d$, $y=2n+2n^2/d$ and $z=2n+d+2n^2/d$ where n runs through all integers while d is in turn all divisors of $2n^2$ less than $n\sqrt{2}$. If a table has been made for $1\leqslant n\leqslant N$ then all $x<2N+3$, all $y<N\cdot(2+\sqrt{2})$ and all $z<N\cdot(2+2\sqrt{2})$ will have appeared which are members of Pythagorean triples. Primitive triples occur when d and $2n^2/d$ are relative prime. (Note that the set of z values is precisely the set of all primes of type $4n+1$ and all their multiples.) The method lends itself smoothly to execution on a programmable pocket calculator; the first 500 Pythagorean triples were obtained in less than one hour.

L. E. Dickson provides an extensive review [**6**] of the sizable literature on Pythagorean triples; in ancient times the emphasis was on getting triples and establishing "rules" for them. Completeness and primitivity were ignored. For instance, the solution (1) was cited by Diophantus (see [**6**]) without t and the relative prime requirement for u and v, but the first proof that (1) is a complete, non-redundant solution is from this century. The present analysis, based on one geometric parameter (running freely through all integers) being twice the area-perimeter ratio and on one auxiliary parameter d, allows an orderly listing in terms of increasing area-perimeter ratio of all Pythagorean triples.

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The Accuracy of Probability Estimates

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Early in a first introduction to probability and statistics one should give some experimental problems in which the student is to empirically estimate probabilities. There are good reasons for problems of this type. First, they introduce the student to one of the most important estimation procedures in statistics—that of estimating probabilities by actual or simulated experimentation. This estimation by simulation is becoming increasingly important as students have their own programmable calculators, which can be used in simulating a variety of random phenomena. Secondly, the student can compare theoretical probabilities with empirical probabilities to see that his probability model fits. Of course, the student needs some idea as to how good his estimates are. It has been my experience that the student (or the general mathematician for that matter) tends to expect estimates to be more accurate than they actually are. The purpose of this note is to give a good, easily stated rule regarding the accuracy of probability estimates. We state this rule as follows:

> *Let A be an event associated with a random experiment ξ. Suppose we make N independent trials of ξ and A occurs n times in these N trials. If we estimate $p = Prob(A)$ by $\hat{p} = n/N$ then at least* 91.0% *of the time* (i.e., *with probability at least* .910) *$\hat{p}$ will be within $1/\sqrt{N}$ of p for any N and p.*

Of course, the same rule can be used when one estimates a binomial proportion p by the proportion of success $\hat{p}$ which occur in N trials. In the following discussion we see that 91.0% is the best overall percentage that one can achieve.

Tchebyshev's Inequality tells us for each integer N, that $\hat{p} \pm 1/\sqrt{N}$ will be at least a 75% confidence interval for p. Of course, the Central Limit Theorem tells us that for large N and p close to 1/2, $\hat{p} \pm 1/\sqrt{N}$ is about a 95% confidence interval for p. Thus the best overall percentage is somewhere between 75 and 95%. In preliminary numerical investigations it appeared that the lowest percentage occurred when $N=6$. We were later able to show that the worst case was indeed $N=6$ with p approaching $5/6 - 1/\sqrt{6} = 0.425085\ldots$ from the left. Here the percentage dips to 91.011%. Thus the

91.0% in our rule is the best overall percentage one can achieve. To prove the rule, we use the (very crude) lower bound given by Tchebyshev's Inequality to handle small p, the normal approximation to the binomial for moderate size p and large N, and numerical techniques to handle the other cases. The details follow.

Proof. It is easy to see that one need only consider $p \leqslant 1/2$. Let $q = 1-p$ as usual. Tchebyshev's Inequality says that $f_N(p) = f(p) = \text{Prob}(|\hat{p}-p| \leqslant 1/\sqrt{N}) \geqslant 1-pq$. Now $1-pq \geqslant 0.91$ if $p \leqslant 0.1$. Let us then consider p in $I=[0.1,0.5]$. For p in this range and $N \geqslant 196$, $Np > 19.6$ and $0 < Np \pm 2\sqrt{Npq} < N$. Thus according to the usual rules of thumb, the normal approximation to the binomial should be adequate. According to Raff ([**1**], p. 295), the error in the normal approximation to the sum of any number of consecutive terms of the binomial density is at most $0.140/\sqrt{Npq}$. Thus for $N \geqslant 196$ and p in I, the error is less than 0.034. Now the normal approximation to $f(p)$ is

$$\phi\left(\left(\lfloor Np+\sqrt{N}\rfloor + 0.5 - Np\right)/\sqrt{Npq}\right) - \phi\left(\left(\lceil Np-\sqrt{N}\rceil - 0.5 - Np\right)/\sqrt{Npq}\right) \tag{1}$$

where ϕ is the distribution function of the standard normal, $\lfloor x \rfloor$ is the greatest integer in x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x. The normal approximation in (1) is at least

$$\phi((\sqrt{N}-0.5)/\sqrt{Npq}) - \phi((-\sqrt{N}+0.5)/\sqrt{Npq}) \geqslant \phi(1.92) - \phi(-1.92) \approx 0.9451 \quad \text{for} \quad N \geqslant 196.$$

Thus $f(p) \geqslant 0.9451 - 0.034 \geqslant 0.9111$ for p in I and $N \geqslant 196$.

Finally, we consider p in I and $1 \leqslant N < 196$. For fixed N and p_0, let us consider all confidence intervals of the type $\hat{p} \pm 1/\sqrt{N}$ which contain p_0. The intersection of all such intervals will be an interval $[a,b]$ where

$$a=0 \quad \text{or} \quad a = x/N + 1/\sqrt{N}, \qquad \text{some integer } x \text{ in } [0,N]$$

and

$$b=1 \quad \text{or} \quad b = y/N - 1/\sqrt{N}, \qquad \text{some integer } y \text{ in } [0,N].$$

For simplicity assume the second form in both cases. Then

$$f(p) = \sum_{k=x}^{y} \binom{N}{k} p^k q^{N-k}$$

for all p in $[a,b]$. Now we can cover I by intervals of the above type and $f(p)$ is uniformly continuous on each of these intervals. Thus it should be possible to find a good lower bound to $f(p)$ over I using numerical techniques. The simplest way to do this would be to find the minimum value of $f(p)$ over a sufficiently fine partition. The problem is to find a partition which will produce the desired accuracy without using too much computer time. In this direction we see that

$$f'(p) = \sum_{x}^{y} (k-Np)\binom{N}{k} p^{k-1} q^{N-k-1}.$$

Then

$$\begin{aligned} |f'(p)| &\leqslant \sum_{k=0}^{N} |k-Np| \binom{N}{k} p^{k-1} q^{N-k-1} \\ &\leqslant \left(\sum_{k=0}^{N} |k-Np|^2 \binom{N}{k} p^k q^{N-k} \right)^{1/2} \Big/ pq \\ &= \sqrt{N/pq} \leqslant \sqrt{N}/0.3 \quad \text{for} \quad p \text{ in } I. \end{aligned}$$

Thus we have a bound on $f'(p)$ which does not depend on x or y. Hence if p and p' are in the same interval $[a,b]$ and $|p-p'| \leqslant 0.3\varepsilon/\sqrt{N} = d$ then $|f(p)-f(p')| \leqslant \varepsilon$. If we take a partition π of $[0.1,0.5]$ with points separated by the distance d and if all intervals of type $[a,b]$ above contain partition points for the fixed N then

$$\min\{f(p) \mid 0.1 \leqslant p \leqslant 0.5\} \geqslant \min\{f(p) \mid p \in \pi\} - \varepsilon.$$

Using choices of ε (depending on N) in the range 0.001 to 0.039, we programmed the above computations on an IBM 360-65. The results showed that $f(p) > 0.91$ for all $N \neq 6$ and p in I. The case $N=6$ was handled analytically since (nearly) 91.0% was attained in this case.

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Platonic Divisions of Space

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Five arrangements of planes in space, which we call the "Platonic" arrangements, are obtained by extending the face-planes of the five Platonic solids. An interesting and natural problem is to determine how many cells these five arrangements form in space.

To set our notation, suppose that n planes meet to form l lines and p points of intersection, C cells, C' of which are bounded (polyhedra), F faces, F' of which are bounded (polygons), and E edges, E' of which are bounded (line segments). In this note we find the following numerical results for the five Platonic arrangements:

Arrangement	n	l	p	C	C'	F	F'	E	E'
Tetrahedron	4	6	4	15	1	28	4	18	6
Cube	6	12	8	27	1	54	6	36	12
Octahedron	8	24	14	59	9	128	32	84	36
Dodecahedron	12	60	52	185	63	432	192	300	180
Icosahedron	20	180	274	835	473	2060	1340	1500	1140

The results for the tetrahedral and cubical arrangements can be obtained visually, and the octahedral arrangement can be studied with just a little more effort. The other two arrangements are significantly more complicated and seem to require a more systematic study.

After some preliminaries about simple arrangements of lines in the plane, we use the elegant sweep-plane method of Alfred Brousseau [3] to establish general formulas for C, C', F, F', E, and E' for arrangements of planes that have no multiple lines. Our numerical results for the Platonic arrangements are obtained from these general formulas. Brousseau has written that he has also obtained some of these numerical results.

An arrangement (i.e., finite set) of l lines in the plane is **simple** if no three of the lines are concurrent. Parallels are permitted; we do not demand that each two lines intersect. To avoid trivial special cases, we suppose throughout that $l \geq 2$, although some of the formulas work also for $l=1$. Formula (1) of the following lemma is due to Brousseau [3].

LEMMA. *If the $l \geqslant 2$ lines of a simple arrangement meet to form R regions, R′ of which are bounded, S segments and rays, S′ of which are bounded (i.e., segments), and p points, then*

$$R = 1 + l + p \tag{1}$$

$$S = l + 2p \tag{2}$$

and, if the l lines are not all parallel,

$$R' = 1 - l + p \tag{3}$$

$$S' = -l + 2p. \tag{4}$$

Proof. Although easy proofs of (1) and (2) can be given by induction on l, and other short arguments using Euler's formula are possible, we use Brousseau's ingenious sweep-line argument. Let b be a line not parallel to any of the lines of the arrangement that is so far away that all the points of intersection and all the bounded regions and segments are to one side. In this initial position, b meets the l lines of the arrangement in l distinct points. Consequently, the sweep-line initially identifies $1 + l$ regions, one for each of the $1 + l$ pieces into which the sweep-line itself is divided by the l points of intersection, and l rays, one for each of the l points of intersection.

Now sweep the line b parallel to itself across the arrangement. New regions, segments, and rays are encountered precisely at the points determined by the arrangement; and because the arrangement is simple, exactly one new region and two new segments or rays appear at each point. Since there are p points, formulas (1) and (2) follow at once.

If the l lines are not all parallel, then each line is crossed by another line, and $2l$ rays are formed. A circle that surrounds all the points of intersection is divided into $2l$ arcs by these $2l$ rays, and each arc lies in a well-defined unbounded region. Consequently, there are $2l$ unbounded regions, and formulas (3) and (4) follow by subtraction from (1) and (2).

Since $k \geqslant 2$ lines in general position (no two parallel and no three concurrent) determine $\binom{k}{2}$ points of intersection, the following very well-known formulas are immediate consequences of the lemma:

$$R = 1 + k + \binom{k}{2}, \qquad S = k + 2\binom{k}{2} = k^2,$$

$$R' = 1 - k + \binom{k}{2} = \binom{k-1}{2}, \qquad S' = -k + 2\binom{k}{2} = k(k-2).$$

As an example, consider the arrangement of $l = n$ lines in the plane generated by extending the n sides of a regular n-gon. (These arrangements are the plane analogs of the Platonic arrangements in space.) It is clear that for each n-gon, the associated arrangement is simple. When n is odd, each two lines meet, and so the lines are in general position. Consequently, R, S, R', and S' are given by the above formulas with $k = n$. Opposite sides of the n-gon are parallel when n is even, but every other two lines intersect. Thus $p = \binom{n}{2} - n/2$, and according to the lemma,

$$R = 1 + \frac{n}{2} + \binom{n}{2}, \qquad S = n(n-1),$$

$$R' = 1 - \frac{3n}{2} + \binom{n}{2}, \qquad S' = n(n-3).$$

Further examples appear in [**6**]. Arbitrary arrangements of lines are studied in [**2**].

Now we are ready to consider arrangements in space. An arrangement (finite set) of planes in space is called **line-simple** if there are no multiple lines, i.e., if no line lies in more than two of the planes. Other kinds of degeneracies—multiple points, parallel planes, and parallel lines of intersection—are all allowed.

Let $\mathcal{A}$ be a line-simple arrangement of n planes in space. To avoid special cases we assume that $n \geqslant 3$ throughout, although some of the formulas also work for $n=1$ or $n=2$. For each point P, let $\lambda(P)$ be the number of planes of $\mathcal{A}$ that pass through P. If $\lambda(P) \geqslant 3$ we say that P is **determined** by $\mathcal{A}$ and has **multiplicity** $\lambda(P)$. Let $\mathcal{P}$ be the set of points determined by $\mathcal{A}$. An arrangement $\mathcal{A}$ is called **planar** if there is a line to which all the planes of $\mathcal{A}$ are parallel (i.e., if the normal vectors to the planes of $\mathcal{A}$ are coplanar). Such an arrangement plainly forms as many unbounded cells, faces, and edges as there are regions, segments and rays, and points, respectively, in its cross section, and it forms no bounded cells, faces, or edges at all.

THEOREM. *Suppose the $n \geqslant 3$ planes of a line-simple arrangement $\mathcal{A}$ meet to form l lines. Then they determine*

$$C = 1 + n + l + \sum_{P \in \mathcal{P}} \binom{\lambda(P)-1}{2} \text{ cells,} \tag{5}$$

$$F = n + 2l + \sum_{P \in \mathcal{P}} \lambda(P)(\lambda(P)-2) \text{ faces, and} \tag{6}$$

$$E = l + \sum_{P \in \mathcal{P}} \binom{\lambda(P)}{2} \text{ edges;} \tag{7}$$

and if the arrangement is not planar, there are

$$C' = -1 + n - l + \sum_{P \in \mathcal{P}} \binom{\lambda(P)-1}{2} \text{ bounded cells,} \tag{8}$$

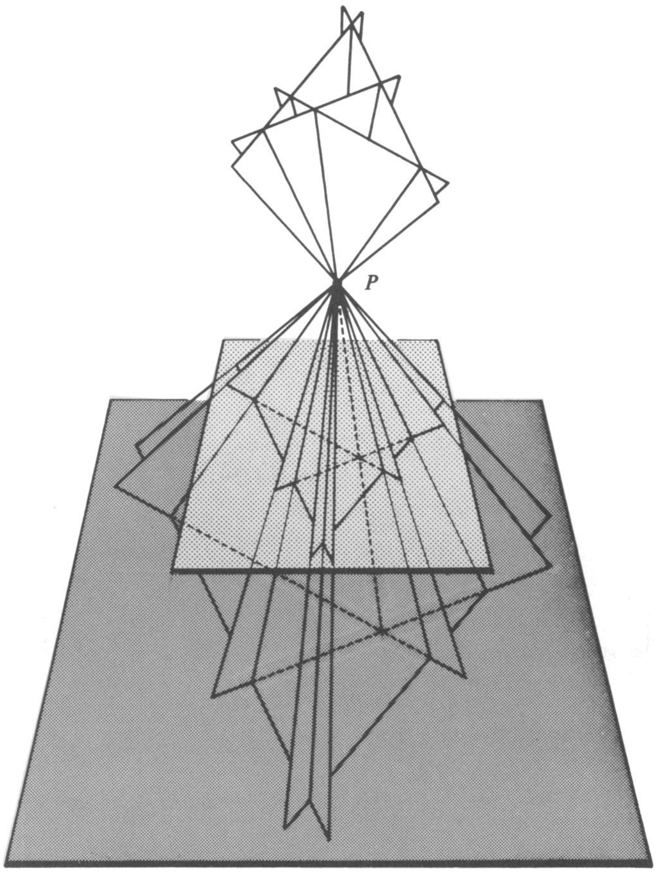
$$F' = n - 2l + \sum_{P \in \mathcal{P}} \lambda(P)(\lambda(P)-2) \text{ bounded faces, and} \tag{9}$$

$$E' = -l + \sum_{P \in \mathcal{P}} \binom{\lambda(P)}{2} \text{ bounded edges.} \tag{10}$$

Proof. Although proofs of the first three formulas can easily be given by induction on n, we use a sweep-plane argument that is analogous to the sweep-line argument we employed to prove the lemma.

Let β be a plane that is not parallel to any of the planes of $\mathcal{A}$ nor to any of the lines in which those planes meet, and suppose that β is so far away that all the points, bounded cells, bounded faces, and segments determined by $\mathcal{A}$ are on the same side. In this initial position, β meets the n planes of $\mathcal{A}$ in a simple arrangement of n lines that intersect to form l points. According to (1) and (2), there are $1+n+l$ regions and $n+2l$ segments and rays formed on β. Consequently, the sweep-plane β initially identifies $1+n+l$ unbounded cells, one for each of the regions formed on β, $n+2l$ unbounded faces, one for each segment and ray on β, and l rays, one for each point on β.

Now sweep the plane β parallel to itself across the arrangement. New cells, faces, and edges are encountered precisely at the points determined by $\mathcal{A}$. A moment's contemplation shows (FIGURE 1) that exactly as many new cells, faces, and edges are encountered at P as there are bounded regions, segments, and points, respectively, formed on β by the $\lambda(P)$ planes through P. These planes meet β in an arrangement of $\lambda(P)$ lines that are in general position. (A multiple point could arise only from a multiple line, and if two lines were parallel on β, the corresponding planes would intersect in a line through P parallel to β, contrary to the initial choice of β.) It follows from the lemma that these $\lambda(P)$ lines form $\binom{\lambda(P)-1}{2}$ bounded regions, $\lambda(P)(\lambda(P)-2)$ line segments, and $\binom{\lambda(P)}{2}$ points; and so the sweep-plane β encounters $\binom{\lambda(P)-1}{2}$ new cells, $\lambda(P)(\lambda(P)-2)$ new faces, and $\binom{\lambda(P)}{2}$ new edges at P. Summing over $\mathcal{P}$ proves formulas (5), (6), and (7).



Sweep-plane rising to a point P of multiplicity four.

FIGURE 1.

The sweep-plane β initially identifies $1+n+l$ unbounded cells, $n+2l$ unbounded faces, and l rays. After β has moved through all the points determined by $\mathcal{Q}$, there are as many *new* unbounded cells, unbounded faces, and rays as there are bounded regions, line segments, and points on β, viz., $1-n+l$, $-n+2l$, and l, provided the arrangement is not planar. (This nice observation is due to G. L. Alexanderson.) So in all there are

$$(1+n+l)+(1-n+l)=2+2l \quad \text{unbounded cells,}$$

$$(n+2l)+(-n+2l)=4l \quad \text{unbounded faces, and}$$

$$(l)+(l)=2l \quad \text{rays}$$

formed by a non-planar line-simple arrangement. Now formulas (8), (9), and (10) follow by subtraction from (5), (6), and (7).

An arbitrary arrangement of planes is **point-simple** if each of the points it determines is of multiplicity three; and an arrangement is **simple** if it is both point-simple and line-simple. A simple arrangement can still have parallel planes and parallel lines of intersection. (Arbitrary arrangement of planes in space are studied in [1].) For simple arrangements the summands occurring in the formulas of the Theorem are constant, so the sums are multiples of the number p of points in the arrangement. The resulting formulas are very pretty:

COROLLARY. *If the arrangement $\mathcal{A}$ is simple, then*

$$C=1+n+l+p,\quad F=n+2l+3p,\quad E=l+3p$$

and, provided the arrangement is not planar,

$$C'=-1+n-l+p,\quad F'=n-2l+3p,\quad E'=-l+3p.$$

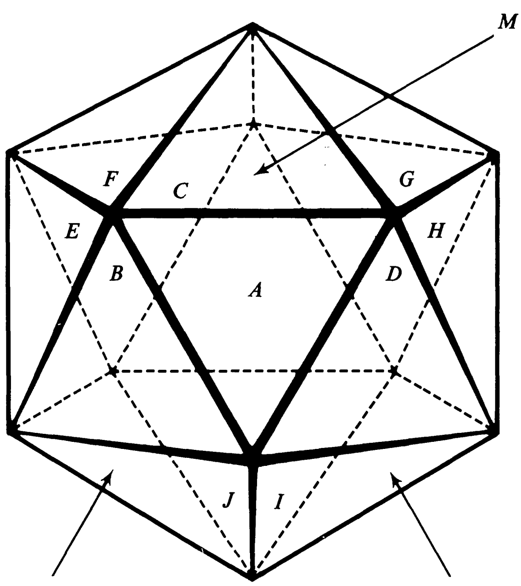
Now we apply these formulas to count the parts formed by the extended face-planes of the Platonic solids. It is evident from convexity that the five Platonic arrangements are line-simple, so the formulas of the Theorem apply.

Tetrahedron. This arrangement of $n=4$ planes is simple, and $l=6$ and $p=4$. According to the formulas of the Corollary, $C=15$, $C'=1$, $F=28$, $F'=4$, $E=18$, and $E'=6$.

Cube. This arrangement of $n=6$ planes is simple, and $l=12$ and $p=8$. The formulas of the Corollary give $C=27$, $C'=1$, $F=54$, $F'=6$, $E=36$, and $E'=12$.

Octahedron. This arrangement of $n=8$ planes is more complicated. Since opposite face-planes are parallel, these eight planes meet to form $l=\binom{8}{2}-4=24$ lines, 12 of which are the extended edges of the octahedron and 12 of which are formed by pairs of alternate face-planes at each vertex. Each of the six vertices is a multiple point of multiplicity four, and there is another point of multiplicity three above each face, formed by the three face-planes that share an edge with that face. One can see that there are no other points of intersection by counting the triples of planes. There are $\binom{8}{3}=56$ such triples. The six vertices of multiplicity four account for $6\binom{4}{3}=24$ triples, the eight points of multiplicity three account for eight more triples, and the four pairs of parallel opposite face-planes account for the remaining 24 triples. So the eight planes determine $p=14$ points. Thus, according to the formulas of the Theorem, $C=59$, $C'=9$, $F=128$, $F'=32$, $E=84$, and $E'=36$.

Dodecahedron. This arrangement has 12 planes that meet to form $l=\binom{12}{2}-6=60$ lines of intersection, 30 of which are extended edges and 30 of which are disjoint from the dodecahedron.



An icosahedron with some faces labeled.

FIGURE 2.

Each of the 20 vertices is a point of multiplicity three. The five planes of the faces that surround each face are concurrent in a point of multiplicity five located above that face, and over each vertex there is a point of multiplicity three, formed by the planes of the three faces that are joined to that vertex by an edge. Nice illustrations that show these points can be found in Holden [**5**, pp. 84–87].

That no other points are determined by these 12 planes can again be verified by counting the triples of face-planes. There are $\binom{12}{3}=220$ such triples. The vertices account for 20 triples, the 12 points of multiplicity five over the faces account for $12\binom{5}{3}=120$ triples, the 20 points of multiplicity three over the vertices account for 20 more triples, and the remaining 60 triples are accounted for by the fact that opposite face-planes are parallel. So these 12 planes determine $p=52$ points. Hence the formulas of the Theorem now give $C=185$, $C'=63$, $F=432$, $F'=192$, $E=300$, and $E'=180$.

Icosahedron. This arrangement has $n=20$ planes that meet to form $l=\binom{20}{2}-10=180$ lines of intersection, 30 of which are extended edges and 150 of which are disjoint from the icosahedron.

To describe the points of intersection, we name the faces of the icosahedron as shown in FIGURE 2. There are $p=274$ points of various multiplicities, as described in the following table. These points are pictured in the drawings of the 59 stellations of the icosahedron given by Coxeter et al. in [**4**].

Typical Planes	Multiplicity	Total Count	Triples
BCD	3	20	$20\binom{3}{3}=20$
EFGHIJ	6	20	$20\binom{6}{3}=400$
BCM	3	60	$60\binom{3}{3}=60$
DEF	3	60	$60\binom{3}{3}=60$
CDEJ	4	30	$30\binom{4}{3}=120$
CEI	3	60	$60\binom{3}{3}=60$
ABDIJ	5	12	$12\binom{5}{3}=120$
CEHKL	5	12	$12\binom{5}{3}=120$
		274	960

That all the points of intersection have been found can be established by counting triples as in the previous cases. There are $\binom{20}{3}=1140$ triples, and 960 of them are accounted for in the table. Since the parallel opposite face-planes account for 180 more triples, we have them all. So, according to the formulas of the Theorem, $C=835$, $C'=473$, $F=2060$, $F'=1340$, $E=1500$, and $E'=1140$.

Pederson [7] gives interesting visualizations of some of the cells associated with the Platonic arrangements based on braided models.

The authors would like to thank Jean Pedersen and Marianne Wojciechowski Jankowski for their assistance with the figures.

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Averaging an Alternating Series

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In calculus and its applications we often encounter alternating series whose terms decrease slowly. The partial sums of such a series may therefore be relatively useless for an accurate numerical estimate of the true sum. A well-known example is the series whose nth partial sum is

$$s_n = 1 - 1/3 + 1/5 - 1/7 + \cdots + (-1)^n/(2n+1)$$

If we take the first 10 terms of the series, the error is .025, more than 3% of the true sum $\pi/4 \cong .78539816$. Since the terms oscillate about the true sum, it is tempting to form the averages $\bar{s}_n = \frac{1}{2}(s_n + s_{n-1}), \bar{\bar{s}}_n = \frac{1}{2}(\bar{s}_n + \bar{s}_{n-1})$ and so forth, in order to speed up the convergence. In order to obtain numerical evidence, we did some calculations with a TI-30 hand calculator:

n	s_n	$\bar{s}_n$	$\bar{\bar{s}}_n$
1	1.00000000	———	———
2	.66666666	.83333333	———
3	.86666666	.76666666	.80000000
4	.72380953	.79523810	.78095238
5	.83492064	.77936509	.78730160
6	.74401155	.78946610	.78441560
7	.82093463	.78247309	.78596960
8	.75426797	.78755130	.78501220
9	.81309150	.78367974	.78561552
10	.76045992	.78677571	.78522773

The table shows us that $|\bar{s}_{10} - \pi/4| \cong .00138411$, an error of less than two-tenths of one percent. A further averaging gives $|\bar{\bar{s}}_{10} - \pi/4| \cong .00017043$, an error of less than three-hundredths of one percent. With repeated averaging (not shown in the table), *we can compute π to an accuracy of 10^{-5} using only the first ten terms of the series.* Clearly this technique deserves further study. (The material presented below complements the article of R. P. Boas **[1]** which mainly deals with series of positive terms.)

In order to fix notation, we first prove a known result which gives a sharp remainder estimate for a large class of alternating series. Let $\{a_n\}_{n\geqslant 1}$ be a sequence which tends monotonically to zero, and set $b_n = a_n - a_{n+1}$, $s_n = a_1 - a_2 + \cdots + (-1)^{n+1}a_n$, and $L = \lim_{n\to\infty} s_n$. Calabrese showed in **[2]** that if $\{b_n\}_{n\geqslant 1}$ also tends monotonically to zero, then $|s_n - L| \leqslant a_n/2$. This follows from two straightforward calculations:

$$\begin{aligned} s_n - L &= (-1)^{n+1}[a_{n+1} - a_{n+2} + \cdots] \\ &= (-1)^{n+1}[(b_{n+1} + b_{n+2} + \cdots) - (b_{n+2} + b_{n+3} + \cdots) + \cdots] \\ &= (-1)^{n+1}[b_{n+1} + b_{n+3} + \cdots], \end{aligned}$$

and

$$a_n = b_n + b_{n+1} + \cdots \geqslant b_{n+1} + b_{n+1} + b_{n+3} + b_{n+3} + \cdots = 2(b_{n+1} + b_{n+3} + \cdots).$$

Therefore $|s_n - L| \leqslant a_n/2$, as required.

We now consider the averages $\bar{s}_n \equiv \frac{1}{2}(s_n + s_{n-1})$ where we define $c_n \equiv b_n - b_{n+1} = a_n - 2a_{n+1} + a_{n+2}$. We assume in this case that the sequence $\{c_n\}_{n\geqslant 1}$ tends monotonically to zero. Then, under these hypotheses, $|\bar{s}_n - L| \leqslant b_{n-1}/4$. We prove this by reducing it to the previous case:

$$\begin{aligned}\bar{s}_n - L &= \tfrac{1}{2}(s_n - L) + \tfrac{1}{2}(s_{n-1} - L)\\ &= \tfrac{1}{2}(-1)^{n+1}[b_{n+1} + b_{n+3} + \cdots] + \tfrac{1}{2}(-1)^n[b_n + b_{n+2} + \cdots]\\ &= \tfrac{1}{2}(-1)^n[b_n - b_{n+1} + \cdots].\end{aligned}$$

This equation expresses the difference $\bar{s}_n - L$ as the remainder of the first $n-1$ terms of the alternating series $\frac{1}{2}(b_1 - b_2 + \cdots)$. Since this series satisfies the hypotheses of the Calabrese result, we conclude that $|\bar{s}_n - L| \leqslant (\frac{1}{2})(\frac{1}{2})b_{n-1}$, which was to be proved.

To study higher averages, we let $d_n \equiv c_n - c_{n+1}$ and $\bar{\bar{s}}_n = \frac{1}{2}(\bar{s}_n + \bar{s}_{n-1})$. Then

$$\begin{aligned}\bar{\bar{s}}_n - L &= \tfrac{1}{2}(\bar{s}_n - L) + \tfrac{1}{2}(\bar{s}_{n-1} - L)\\ &= \tfrac{1}{4}(-1)^n[c_n + c_{n+2} + \cdots] + \tfrac{1}{4}(-1)^{n-1}[c_{n-1} + c_n + \cdots]\\ &= \tfrac{1}{4}(-1)^{n-1}[c_{n-1} - c_n + c_{n+1} - \cdots].\end{aligned}$$

This is the remainder after $n-2$ terms of the alternating series $\frac{1}{4}(c_1 - c_2 + \cdots)$. If $\{d_n\}_{n\geqslant 1}$ tends monotonically to zero, this series satisfies the hypotheses of the Calabrese result. Therefore $|\bar{\bar{s}}_n - L| \leqslant (1/8)c_{n-2}$.

These methods can be used to systematize the ad hoc methods used by Goldsmith in [3] for improving convergence of the series

$$\log 2 = 1 - 1/2 + 1/3 - \cdots$$
$$\pi/4 = 1 - 1/3 + 1/5 - \cdots.$$

If $a_n = 1/n$, the difference sequences become $b_n = 1/n(n+1)$ and $c_n = 2/n(n+1)(n+2)$. Therefore

$$|s_n - \log 2| \leqslant 1/2n$$
$$|\bar{s}_n - \log 2| \leqslant 1/4n(n-1)$$
$$|\bar{\bar{s}}_n - \log 2| \leqslant 1/4n(n-1)(n-2)$$

Thus $n=5$ gives an error for $\bar{\bar{s}}_n$ of .004; $n=10$ gives an error of .0003. Since Goldsmith's error was of the order of magnitude $0(n^{-4})$, we see that his approximation corresponds to an average of the form $\bar{\bar{\bar{s}}}_n$.

If $a_n = 1/(2n-1)$, the difference sequences become $b_n = 2/(2n+1)(2n-1)$ and $c_n = 4/(2n+1)(2n+3)(2n-1)$. Therefore

$$|\bar{\bar{s}}_n - \pi/4| \leqslant 1/2(2n-3)(2n-1)(2n-5).$$

For $n=10$ this yields $|\bar{\bar{s}}_n - \pi/4| \leqslant .000179$, in remarkable agreement with the numerical value .00017043.

The method of averaging can be pursued to obtain a general result which applies to alternating series whose terms are completely monotonic, that is, the terms and all their successive differences tend monotonically to zero. We define $(\delta^0 a)_n = a_n$ for $n \geqslant 1$ and

$$(\delta^k a)_n = (\delta^{k-1}a)_n - (\delta^{k-1}a)_{n+1}, \qquad (n \geqslant 1, k \geqslant 1)$$

and require that $(\delta^k a)_n \geqslant 0$ for all n, $k \geqslant 1$. The partial sums and higher averages are defined by

$$s_n^{(0)} = a_1 - a_2 + \cdots + (-1)^{n+1}a_n \qquad (n \geqslant 1)$$
$$s_n^{(k)} = \frac{1}{2}\left(s_n^{(k-1)} + s_{n-1}^{(k-1)}\right) \qquad (n \geqslant k+1, k \geqslant 1)$$

As usual, let $L = \lim_{n\to\infty} s_n^{(0)}$. By repeating the reasoning used above, we can arrive at the identity

$$s_n^{(k)} - L = \frac{(-1)^{n+k-1}}{2^k}[(\delta^k a)_{n-k+1} - (\delta^k a)_{n-k+2} + \cdots].$$

The expression in brackets is the remainder of an alternating series which satisfies the hypotheses of

Calbrese's theorem (because $\{a_n\}$ is completely monotonic). Therefore

$$|s_n^{(k)} - L| \leqslant (1/2^{k+1})(\delta^k a)_{n-k} \qquad (n \geqslant k+1).$$

This generalizes the previous results established for $k=1$, 2. Even more interesting is the case $k=n-1$:

$$|s_n^{(n-1)} - L| \leqslant (1/2^n)(\delta^{n-1}a)_1.$$

But $a_1 \geqslant a_2 \geqslant \cdots \geqslant 0$, and therefore the differences $(\delta^k a)_n$ are all less than a_1. Hence

$$|s_n^{(n-1)} - L| \leqslant a_1/2^n.$$

Therefore we have proved the following proposition: *For any alternating series of completely monotonic terms, the averages $s_n^{(n-1)}$ converge geometrically fast to the true sum of the series.* In particular, by taking n terms of the series we get n binary places of accuracy.

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Convex Sets and the Hexagonal Lattice

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Let K be a bounded, closed convex region in the Euclidean plane. The **minimal width** of K is the smallest distance between parallel supporting lines of K; we denote this by $\delta(K)$. Also, we denote by Λ_0 the (planar) lattice of points with integral coordinates and by Λ_1 the hexagonal lattice generated by the vectors $a=(2,0)$ and $b=(1,\sqrt{3})$. In [2] it is shown that if $\delta(K) \geqslant \frac{1}{2}(2+\sqrt{3})$, then K contains a point of the integral lattice Λ_0, and this result is best possible. The proof is not long but requires some ingenuity. In contrast, the corresponding result for the hexagonal lattice Λ_1 is delightfully easy to establish. We show:

If $\delta(K) \geqslant 2\sqrt{3}$, then K contains a point of the hexagonal lattice Λ_1. Further, $\delta(K)=2\sqrt{3}$ without K containing a lattice point, if and only if K is the equilateral triangle of side length 4.

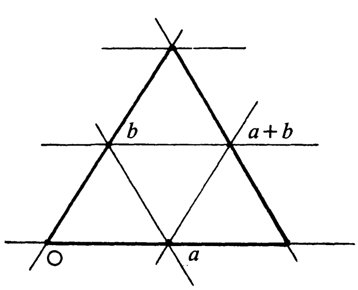

FIGURE 1.

Let K denote a region which contains no lattice points in its interior, and for which the minimal width is as large as possible. The triangle $\triangle$ illustrated in FIGURE 1 shows that for such a region K, $\delta(K) \geqslant 2\sqrt{3}$.

Blaschke [1] proves that any convex figure of minimal width x contains a circular disc of radius $x/3$. Hence K contains a disc D of radius $(2\sqrt{3})/3$. Clearly D contains no points of Λ_1 in its interior. Now the points of Λ_1 occur as the vertices of a tessellation of equilateral triangles of side length 2. Since $(2\sqrt{3})/3$ is the circum-radius of such an equilateral triangle, D must have three points of Λ_1 on its boundary. By suitably translating K we may take these lattice points to be $a, b, a+b$.

Since K is convex and contains no points of Λ_1 in its interior, K is bounded by the tangents to D at $a, b, a+b$; that is, K is bounded by the sides of the triangle $\triangle$. Hence $K \subseteq \triangle$, and $\delta(K) \leqslant \delta(\triangle) = 2\sqrt{3}$.

To complete the proof of our result, we notice that no closed, convex, proper subset of the equilateral triangle has the same minimal width as the triangle. For, no such subset can contain all three vertices of the triangle, and removal of any vertex of the triangle decreases the minimal width.

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A Property of 70

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It is well known (see, e.g., [3]) that 30 is the largest integer with the property that all smaller integers relatively prime to it are primes. In this note I will consider a related situation in which the corresponding special number turns out to be 70. (For a while I believed 30 to be the key figure in the new context,too, but E. G. Straus showed me that the correct value was indeed 70.) Following the proof of this special property of 70, I will mention a few related problems, some of which seem to me to be very difficult. I hope to convince the reader that there are very many interesting and new problems left in what is euphemistically called "elementary" number theory. Although these problems are easy to comprehend, their solutions will undoubtedly require either remarkable ingenuity or extensive application of known techniques.

Throughout this paper we will be studying sequences of positive integers related to a given integer n. The basic sequence $\{a_i\}_{i=0}^{\infty}$ begins with $a_0 = n$; once $a_0, a_1, \ldots, a_{k-1}$ are known, a_k is chosen to be the smallest integer greater than a_{k-1} that is relatively prime to the product $a_0 a_1 \cdots a_{k-1}$. Clearly each prime greater than n is an a_k. Moreover, each a_k greater than n^2 is a prime. Table 1 contains examples of the sequences $\{a_k\}$ corresponding to certain integers n.

Theorem 1. 70 *is the largest integer for which all the* a_k (*for* $k \geqslant 1$) *are primes or powers of primes.*

Proof. I will try to make the proof as short as possible; thus it is not as elementary as it might be. We begin with the difficult but useful result [5] that for $x > 17/2$, there are at least *three* primes in the interval $(x, 2x)$. Hence, for $n > 17^2 = 289$ there are at least three primes in the interval $\left(\frac{1}{2}n^{1/2}, n^{1/2}\right)$. Furthermore, at least one of these primes does not divide n since their product exceeds $n^{3/2}/8$ (which in turn is greater than n). Thus, if p_1 is the greatest prime satisfying $p_1 < n^{1/2}$ and $p_1 \nmid n$ we know that $p_1 > \frac{1}{2}n^{1/2}$. Also, for $n > 289$, there are at least three primes in $(2n^{1/2}, n/4)$ since $n > 16n^{1/2}$. At least one of these three primes does not divide n since their product exceeds $4n$. Hence, if q_1 is the least prime satisfying $q_1 \geqslant n^{1/2}$ and $q_1 \nmid n, q_1 < n/4 < p_1^2$.

n	non-prime a_k	$f(n)$	a_k that are not prime powers
3	2^2	0	
4	3^2	0	
5	$2\cdot 3$	1	6
6	5^2	0	
7	$2^3,3^2,5^2$	0	
8	$3^2,5^2,7^2$	0	
9	$2\cdot 5,7^2$	1	10
10	$3\cdot 7$	1	21
11	$4\cdot 3,5^2,7^2$	1	12
12	$5^2,7^2,11^2$	0	
15	$2^4,7^2,11^2,13^2$	0	
18	$5^2,7^2,11^2,13^2,17^2$	0	
22	$5^2,3^3,7^2,13^2,17^2,19^2$	0	
24	$5^2,7^2,11^2,13^2,17^2,19^2,23^2$	0	
30	$7^2,11^2,13^2,17^2,19^2,23^2,29^2$	0	
31	$2^5,3\cdot 11,5\cdot 7,11^2,13^2,19^2,23^2,29^2$	2	33,35
46	$7^2,3\cdot 17,5\cdot 11,13^2,19^2,29^2,31^2,37^2,41^2,43^2$	2	51,55
70	$9^2,11^2,13^2,17^2,19^2,23^2,29^2,31^2,37^2,41^2,43^2,47^2,53^2,59^2,61^2,67^2$	0	
71	$2^3\cdot 3^2,7\cdot 11,5\cdot 17,13^2,19^2,\ldots,67^2$	3	72,77,85
97	$2\cdot 7^2,3^2\cdot 11,5\cdot 23,13^2,17^2,19^2,29^2,\ldots,97^2$	3	98,99,115
272	$3\cdot 7\cdot 13,5^2\cdot 11,19^2,23^2,29^2,\ldots,271^2$	2	273,275

SAMPLE SEQUENCES generated from integers n by counting upwards from n, omitting every integer that contains a prime factor in common with any previous terms in the sequence. Since every prime larger than n will automatically be included, we record here only the non-prime numbers that occur in the sequences. (No non-primes occur beyond n^2—as observed in the text—so our record terminates before that point.) The column headed "$f(n)$" records the number of members of the sequence that are neither prime nor a power of a prime. Those numbers for which $f(n)=0$ have the property that all members of the sequence are primes or powers of prime; they are 3, 4, 6, 7, 8, 12, 15, 18, 22, 24, 30, 70. It is proved in the accompanying article that no other numbers have this property.

TABLE 1.

Now consider p_1q_1. If it is one of the a_k's, then the property stated in our theorem—namely, that all a_k are primes or powers of primes—is satisfied for $n>289$. If not, then there must be an a_i with $n<a_i<p_1q_1$ and $(a_i,p_1q_1)>1$. We only have to prove that this a_i must have at least two distinct prime factors. If this does *not* hold, then a_i would have to be a power of p_1 or a power of q_1. Clearly it cannot be a power of q_1 since $q_1^2>p_1q_1$. However, it cannot be a power of p_1, either, since $p_1^2<n$ and $p_1^3>p_1q_1$ (because $p_1^2>q_1$). Thus all a_k corresponding to $n>289$ are primes or powers of primes. The same conclusion holds for $70<n\leqslant 289$, and may be verified by direct computation.

By more complicated methods, we can prove the following related result:

THEOREM 2. *For all sufficiently large n, at least one of the a_k's is the product of exactly two distinct primes.*

I shall not give the proof since it is fairly complicated and uses deep results in analytical number theory. Although I was fairly sure that this result held for every n greater than 70, and thus strengthened Theorem 1, I could not prove this. Recently C. Pomerance found a proof of Theorem 2 for n greater than 6000; he also observed that the result fails for $n=272$ (see TABLE 1).

The following conjecture, related to Theorem 2, seems very difficult. Denote by $p(x)$, the least prime factor of x. Then for sufficiently large n, there are always composite numbers x satisfying

$$n<x<n+p(x) \tag{1}$$

The inequality (1) is a slight modification of an old conjecture that J. L. Selfridge and I proposed in

[2]. In fact, I expect that for sufficiently large n there are squarefree x's satisfying (1) which have exactly k distinct prime factors. I am sure that this conjecture is very deep. It would of course imply Theorem 2 since the integers x satisfying (1) must be a_k's corresponding to the given value of n.

I wish now to state a few simple facts and pose some difficult problems about our a_k's. We have already noted that each prime greater than n is an a_k, and that each a_k greater than n^2 is prime. Let p be a prime less than n and let $a(n,p)$ be the least a_k which is a multiple of p. It is easy to see that $a(n,p) \leqslant p^{\alpha+1}$ where $p^{\alpha} \leqslant n < p^{\alpha+1}$, for if none of the $a_k < p^{\alpha+1}$ are multiples of p, then $p^{\alpha+1}$ is an a_k.

Denote by $f(n)$ the number of those a_k which are not powers of primes. Clearly, $f(n) \leqslant \pi(n^{1/2})$ (where $\pi(x)$ denotes the number of primes $\leqslant x$) since each such a_k must have a prime factor exceeding $n^{1/2}$. (If $n^{1/2} < p$, then p^2 is an a_i and so no a_k can equal pt for $p \leqslant t$). I do not have any good upper or lower bounds for $f(n)$. I conjecture that $f(n) > n^{1/2-\varepsilon}$ for $n > n_0(\varepsilon)$. I am not sure whether $\lim_{n\to\infty} f(n)/\pi(n^{1/2}) = 0$.

Denote by $P(n)$ the largest prime which is less than n. It is not difficult to show that the largest a_k which is not a prime is just $P^2(n)$. On the other hand, I cannot determine the largest a_k which is not a power of a prime. In fact, I cannot even get an asymptotic formula for it and, in fact, have no guess as to its order of magnitude. It may be true that if $n \geqslant n_0(\varepsilon)$ and a_k is not a power of a prime, then $a_k < (1+\varepsilon)n$. Pomerance informs me that he can prove this, and in fact Penney, Pomerance and I are writing a longer joint paper on this subject.

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Rencontre as an Odd-Even Game

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In [3], Schuster and Philippou looked at some nonintuitive aspects of four examples of games that end with either an odd or an even integer. Their analysis of how one should bet in these "odd-even" games is simplified by the statistical independence which is inherent in the Bernoulli and Poisson probability models. We consider here an interesting variation of what is sometimes called the Matching Problem as an example of an odd-even game based on sampling without replacement. The Matching Problem was first published in 1708 by Montmort under the name "*Treize*", later became known as "*Rencontre*", and can be stated as follows:

> Two equivalent decks of different cards are each put into random order and then compared against each other. If a card occupies the same position in both decks, then a match has occurred. What is the probability of no matches?

The game which we want to investigate is played in this way:

> The house shuffles each of the two decks of cards and places them before two players who then alternate selecting a pair of cards, one card from each deck. The first player to choose a matching pair wins the game, whereas the house wins if no match is made.

Assume the two decks each contain n cards. Let $p_1(n)$, $p_2(n)$, and $p_3(n)$ be the respective probabilities of winning for the player who selects first, for the other player, and for the house. In more prosaic, mathematical terms, these are the probabilities that a permutation, randomly chosen from the collection of all $n!$ permutations of the integers 1 through n, first leaves fixed an odd integer, an even integer, or no integer at all. The solution to the Matching Problem (see, e.g., [2]) is surprising in that $p_3(n)$, the probability that no matches are made, is very nearly constant. More precisely,

$$p_3(n)=1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\cdots+(-1)^n\frac{1}{n!}, \tag{1}$$

which is a partial sum of the Maclaurin series expansion for $\exp(x)$ when $x=-1$. Because of the alternating signs, this sum differs from $1/e=.367879\ldots$ by less than $1/(n+1)!$, and

$$p_3(2n)>p_3(2n+2)>1/e>p_3(2n+1)>p_3(2n-1) \tag{2}$$

for $n=1,2,3,\ldots$. Obviously, the house should use small decks containing an even number of cards. However, as can be seen in TABLE 1, it makes little practical difference what size decks are used if there are to be more than, say, a half-dozen cards in each.

n	$p_3(n)$	$p_2(n)$	$p_1(n)$	$\hat{p}_1(n)$
1	0.0	0.0	1.0	0.5
2	0.5	0.0	0.5	0.4410602794
3	0.3333333333	0.1666666667	0.5	0.4508835662
4	0.375	0.25	0.375	0.3724207252
5	0.3666666667	0.2416666667	0.3916666667	0.3900902961
6	0.3680555556	0.2833333333	0.3486111111	0.3485678041
7	0.3678571429	0.2648809524	0.3672619048	0.3672350826
8	0.3678819444	0.2931051587	0.3390128968	0.3390124721
9	0.3678791887	0.27678847	0.3553323413	0.3553320524
10	0.3678794643	0.2982936508	0.3338268849	0.333826882
11	0.3678794392	0.2841645372	0.3479560235	0.3479560214
12	0.3678794413	0.3015621117	0.330558447	0.330558447
13	0.3678794412	0.2891972837	0.3429232751	0.3429232751
14	0.3678794412	0.3038130196	0.3283075392	0.3283075392
15	0.3678794412	0.2928540467	0.3392665121	0.3392665121

TABLE 1.

In TABLE 1 we also see the pattern for the players' chances. The player who goes first always has the advantage over the second player but will never have an advantage over the house when the deck contains six or more cards. Although no simple explicit representation like (1) for either $p_1(n)$ or $p_2(n)$ seems to exist, we can find recursive formulas to calculate these probabilities and to establish convergent oscillatory behavior similar to (2).

THEOREM. *Let* $p=\frac{1}{2}(1-1/e)=0.316060\ldots$. *Then*

$$\lim_{n\to\infty} p_1(n)=\lim_{n\to\infty} p_2(n)=p \tag{3}$$

and

$$p_1(2n-1)>p_1(2n+1)>p_1(2n)>p_1(2n+2)>p>p_2(2n+2)>p_2(2n)>p_2(2n+1)>p_2(2n-1)$$

for $n>1$. (4)

Proof. The convergence in (3) is an easy consequence of Lemma 1 (below) upon noticing that $q_3(n)$ converges to $2p$ by (1) and that $p_2(n)$ equals $q_3(n)-p_1(n)$ by (5) below. The content of Lemma 2 and its use in the proof of the theorem can be seen by referring to FIGURE 1. The top diagram there represents the magnitude of the oscillations of $p_3(n)$ about $1/e$ as given by (1). Bounds for the corresponding oscillations of $p_1(n)$ from the inequalities of Lemma 2 are indicated in the middle diagram which, together with

$$p_1(n)+p_2(n)+p_3(n)=1, \tag{5}$$

yields the bottom diagram and proves (4).

LEMMA 1. *Define* $q_3(n)=1-p_3(n)$. *Then* $p_1(1)=1$ *and*

$$2p_1(2n)=\frac{1}{2n}p_1(2n-1)+q_3(2n) \tag{6}$$

$$2p_1(2n+1)=\frac{1}{2n+1}[p_1(2n)+p_3(2n)]+q_3(2n+1) \tag{7}$$

for $n=1,2,3,\ldots$.

Proof. Let $P(m,n)$ be the number of permutations of the first n natural numbers for which m is the first integer whose position is left unchanged. Then $P(m,n)/n!$ is the probability that, for cards drawn from decks of size n, the first match occurs on the mth draw. The sum of $P(m,n)/n!$ over odd values of m is just $p_1(n)$. The elegant solution to *Rencontre* in **[4]** (see also **[1]**) proves what is apparent in TABLE 2, namely that $P(1,n)=(n-1)!$ and

$$P(m,n)-P(m+1,n)=P(m,n-1) \tag{8}$$

for $m=1,2,3,\ldots,n-1$. Divide the two terms on the left side of (8) by $n!$ and the term on the right side by $n[(n-1)!]$; then sum over odd m from 1 to $n-1$. What you have is $p_1(n)-p_2(n)=\frac{1}{n}p_1(n-1)$ when n is even. However, when n is odd, the sum for $p_1(n)$ on the left side lacks the one term $P(n,n)/n!$. As an exercise, the reader should convince himself that this value equals $p_3(n-1)/n$. Hence $p_1(n)-p_2(n)=\frac{1}{n}[p_1(n-1)+p_3(n-1)]$ when n is odd. Substituting $p_2(n)=q_3(n)-p_1(n)$ from (5) into these last two equations proves (6) and (7) respectively.

$P(m,n)$								
m \ n	1	2	3	4	5	6	7	8
1	1	1	2	6	24	120	720	5040
2		0	1	4	18	96	600	4320
3			1	3	14	78	504	3720
4				2	11	64	426	3216
5					9	53	362	2790
6						44	309	2428
7							265	2119
8								1854

TABLE 2.

LEMMA 2. *For* $n>1$,

$$p_1(2n-1)-p_1(2n+1)>1/(2n)! \tag{9}$$

$$p_1(2n)-p_1(2n+2)>0 \tag{10}$$

$$p_1(2n-1)-p_1(2n)>1/(2n)! \tag{11}$$

$$p_1(2n+1)-p_1(2n)>1/(2n+1)!. \tag{12}$$

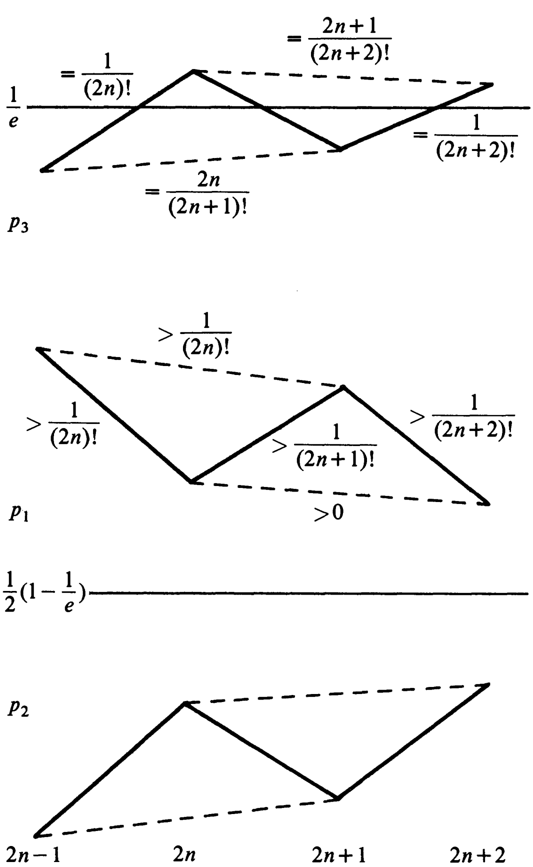
Proof. The reader should check TABLE 1 to satisfy himself that both (9) and (10) are true for the first few values of n. Now assume they are true for some fixed n. To prove (9) for $n+1$ we need to bound $2p_1(2n+1)-2p_1(2n+3)$ below by $2/(2n+2)!$. From (7), this difference equals

$$\left[\frac{p_1(2n)}{2n+1}-\frac{p_1(2n+3)}{2n+3}\right]+\left[\frac{p_3(2n)}{2n+1}-\frac{p_3(2n+2)}{2n+3}\right]+\left[q_3(2n+1)-q_3(2n+3)\right].$$

Now the quantity in the first brackets is greater than $[p_1(2n)-p_1(2n+2)]/(2n+3)$ which is strictly positive by the assumed relation (10). The last quantity equals $(2n+2)/(2n+3)!$ by (1). Using (1) again, we replace $p_3(2n+2)$ with $p_3(2n)-1/(2n+1)!+1/(2n+2)!$ in the second brackets. That makes that quantity equal to $2p_3(2n)/(2n+1)(2n+3)+(2n+1)/(2n+3)!$ which exceeds $(2n+4)/(2n+3)!$ when $p_3(2n)$ exceeds $3(2n+1)/2(2n+2)!$, which is always the case since $p_3(2n)>1/(2n)!$ by (1). The proof of (10) for $(n+1)$ is similar and simpler. From (6) we have that

$$2p_1(2n+2)-2p_1(2n+4)=\left[\frac{p_1(2n+1)}{2n+2}-\frac{p_1(2n+3)}{2n+4}\right]+[q_3(2n+2)-q_3(2n+4)]$$

which, by what has just been proven for (9) and by (1), is strictly greater than $1/(2n+4)(2n+2)!-(2n+3)/(2n+4)!$ which is zero. This proves (9) and (10) simultaneously by induction.



Representations of the oscillations of the probabilities $p_1(n)$, $p_2(n)$, and $p_3(n)$. The formulas by the graphs represent the magnitudes of the vertical displacements of the lines that they are near.

FIGURE 1.

One can check in TABLE 1 to see that (11) and (12) are true for n equal to 2 through 6. Notice there that

$$\frac{2n+1}{2n}p_1(2n-1)<1/e<p_3(2n)$$

for $n=7$. From (9) we know that $p_1(2n-1)$ is a decreasing sequence in n, while $p_3(2n)$ decreases to $1/e$ by (1). Hence the last inequalities remain true and

$$\frac{1}{2n}p_1(2n-1)<\frac{1}{2n+1}p_3(2n)$$

for $n\geqslant 7$. Also $p_1(2n)>1/(2n)!$ for all $n>1$ since $p_1(2n)$ decreases to p, by (10) and (3). These last two inequalities allow us to write

$$\frac{[p_1(2n)+p_3(2n)]}{2n+1}-\frac{p_1(2n-1)}{2n}>\frac{1}{(2n+1)!}$$

Now add $q_3(2n+1)-q_3(2n)$ to the left side and add its equal, $1/(2n+1)!$, to the right side; then divide both sides by 2. What is left is precisely (12) by (6) and (7). And, as FIGURE 1 clearly shows, (12) and (9) together imply (11).

As indicated by (8) in the proof of Lemma 1, the values for $P(m,n)$ in TABLE 2 can be determined simply by placing $(n-1)!$ at the top of the nth column and computing successive entries by subtraction. Moreover, $p_1(n)$ in TABLE 1 can be calculated as the sum of $P(m,n)/n!$ over odd m, and $p_2(n)$ as a similar sum over even m. However, in practice these computations are formidable due to the size of the factorials involved. By using $1/e$ in place of the p_3 sequence in (6) and (7), we can recursively define and compute a new sequence $\hat{p}_1(n)$ starting with $\hat{p}_1(1)=1/2$; see TABLE 1. Since $p_3(n)$ differs from $1/e$ by less than $1/(n+1)!$, it can easily be shown by induction that $p_1(n)$ and $\hat{p}_1(n)$ differ by $1/n!$ at most. In addition to being easier to compute, $\hat{p}_1(n)$ also provides us with an asymptotic explicit formula for $p_1(n)$. One can show from (6) and (7) that the generating function $\sum_{n=1}^{\infty}\hat{p}_1(n)s^n$ satisfies the differential equation

$$2\frac{dy}{ds}-y=(1-1/e)\frac{1}{(1-s)^2}+(1/e)\frac{1}{1-s^2}$$

which, using standard techniques, yields the solution

$$\hat{p}_1(n)=\frac{e-1}{2e}\sum_{i+j+k+1=n}\frac{(-1)^j(k+1)}{2^{i+j}i!j!(j+k+1)}+\frac{1}{2e}\sum_{i+j+2k+1=n}\frac{(-1)^j}{2^{i+j}i!j!(j+2k+1)}.$$

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Square Roots of -1

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It is well known that the equation $x^2+1\equiv 0 \pmod p$, for p an odd prime, is solvable if and only if $p\equiv 1 \pmod 4$. Standard proofs of this result appeal to Fermat's "Little" Theorem and Wilson's Theorem. (See, for example **[1]**, pp. 104–106.) The purpose of this note is to give a direct and somewhat more natural proof.

Writing the equation in the equivalent form

$$x(-x)\equiv 1 \pmod p \tag{1}$$

shows that we seek an element x in the field J_p with the property that its additive inverse serves as its multiplicative inverse as well. Since neither 0 nor ± 1 can have this property and since $-x\equiv p-x$ (mod p), we can restrict our search for solutions to (1) (for $p>3$) to the list $L=\{\pm 2, \pm 3, \ldots, \pm (p-1)/2\}$. (The reader should verify that equation (1) is unsolvable for $p=3$.)

Let S and T be the subsets of L defined by $S=\{2,3,\ldots,(p-1)/2\}$ and $T=\{-(p-1)/2,\ldots,-3,-2\}$. Two results are now easy. First, there can be at most one $s\in S$ satisfying equation (1), for $s_i(-s_i)\equiv 1$ (mod p), $i=1,2$, implies that $s_1^2\equiv s_2^2$ (mod p) or $(s_1-s_2)(s_1+s_2)\equiv 0$ (mod p). Since $0<s_1+s_2<p$, we have $s_1\equiv s_2 \pmod p$. But $s_1,s_2\in S$; hence, $s_1=s_2$. Second, if $s_1,s_2\in S$ satisfy $s_1s_2\equiv 1$ (mod p), then $(-s_1)(-s_2)\equiv 1 \pmod p$, and if $s(-t)\equiv 1 \pmod p$, $t\neq s$, then $t(-s)\equiv 1 \pmod p$.

Now, if there is a solution $s\in S$ to equation (1), then it follows from the first observation that there is exactly one such solution, and we strike $\pm s$ from L. By the second result the remaining elements in L can be stricken in groups of 4: if $s_1s_2\equiv 1 \pmod p$, strike $\pm s_i, i=1,2$, and if $s(-t)\equiv 1 \pmod p$, $s\neq t$, strike $\pm s, \pm t$. Since $|L|\equiv 2$ (mod 4), it follows that $p\equiv 1$ (mod 4).

On the other hand, if $p\equiv 1$ (mod 4), then $|L|\equiv 2$ (mod 4) and after the quadruples of non-solutions have been stricken according to the second observation, we must have two solutions left over, one in S and the other in T. This finishes the proof of our result.

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PROBLEMS

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Proposals

To be considered for publication, solutions should be mailed before April 1, 1979.

1048. Let $\{a_k\}$ be an increasing sequence of positive integers with $a_{k+1}/a_k \to 1$ as $k \to \infty$. Prove that $\sum_{k=1}^{\infty}(a_k-1)^2/a_1 \cdots a_k$ is irrational. What happens if it is not assumed that $a_{k+1}/a_k \to 1$ but the series converges? I have some partial results but no complete discussion. [*Paul Erdös, Hungarian Academy of Science.*]

1049. For nonnegative integers n, let $L_n = \binom{2n}{n}/(n+1)$. Prove that $\sum_{k=0}^{n} L_k L_{n-k} = L_{n+1}$. [*Edward T. H. Wang, Wilfrid Laurier University.*]

1050. Consider the differential equation $y'' + P_1(x)y' + P_2(x)y = 0$, where P_1 and P_2 are polynomials not both constant. Show that this equation has at most one solution of the form $x^a e^{mx}$ for real a. [*W. R. Utz, University of Missouri, Columbia.*]

1051. A game involves a quizmaster and two players, X and Y. The quizmaster chooses an ordered pair of real numbers (x,y) and tells x to player X and y to player Y. The quizmaster also tells the players that (x,y) is in the set $A = \{(x_i,y_i): i = 1,2,\ldots,n\}$. The quizmaster then asks X and Y alternately if they know (x,y). Find a characterization of the set A which guarantees that either X or Y will eventually know (x,y). [*A. K. Austin, The University of Sheffield.*]

1052. Show that Boolean rings (idempotent commutative rings with identity) are isomorphic if their multiplicative semigroups are isomorphic. [*F. David Hammer, Santa Cruz, California.*]

1053. Let $f(x)$ be differentiable on $[0,1]$ with $f(0) = 0$ and $f(1) = 1$. For each positive integer n, show that there exist distinct $x_1, x_2, \ldots, x_n$ such that $\sum_{i=1}^{n} 1/f'(x_i) = n$. [*Peter Ørno, The Ohio State University.*]

ASSISTANT EDITORS: DON BONAR, *Denison University*; WILLIAM A. MCWORTER, JR., *The Ohio State University. We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to Dan Eustice, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q655. If a, b, c, and d are positive numbers, prove that

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b}\geqslant 2.$$

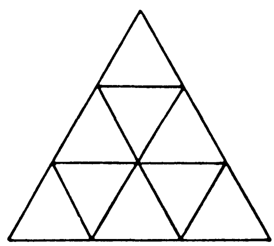
When does equality hold? [*Mark Kleiman, Stuyvesant High School, New York, N.Y.*]

Solutions

Parallelograms **November 1976**

1001

In the accompanying figure, n, the length of the base is 3 units, and $f(n)$, the number of parallelograms, is 15. Find a formula for $f(n)$. (Cf. Problem 889, January, 1974, and Problem 975, March, 1976.) [*Edward T. H. Wang, Wilfrid Laurier University.*]



Solution: For each parallelogram there exists a unique pair of points (not on the same line segment in the triangle) which are the acute vertices of the parallelogram. Conversely, any two points (not on the same line segment in the triangle) are the acute vertices of a unique parallelogram.

Thus $f(n)$ = (the number of pairs of points not on the same line segment of the triangle) = (the number of pairs of points) $-(3)x$ (the number of pairs of points on any horizontal segment of the triangle): Thus,

$$f(n)=\binom{(n+1)(n+2)/2}{2}-3\sum_{k=1}^{n}\binom{k+1}{2}=3\binom{n+2}{4}.$$

Jordan I. Levy, Junior
University of Delaware

Also solved by Steven Alexander, Philip Benjamin, Clayton W. Dodge, Michael Ecker, Richard A. Gibbs, Landy Godbold, Thomas M. Green & Charles L. Hansberg, M. G. Greening (Australia), Karl W. Heuer, Dinh Thê Hung, Eli L. Isaacson, Mark Kleiman, Frederick M. Liss, Janice A. McGodnick, William Moser, Larry O. Olson, John Oman, J. Rue, Wayne Schreiner, Richard Solakiewicz, R. S. Stacy, Pambuccian Victor (Romania), G. Wedderburn, and the proposer.

1002. a. For which values of n is it possible to find a permutation $[a_1,a_2,\dots,a_n]$ of $[0,1,\dots,n-1]$ so that the partial sums $\Sigma_{i=1}^{k}a_i$, $k=1,2,\dots,n$, when reduced modulo n, are also a permutation of $[0,1,\dots,n-1]$? [*Bernardo Recamán, University of Warwick.*]

b.* Find the number of permutations of $[0,1,\dots,n-1]$ for $n \leqslant 12$ which solve part a. Can a general formula for the number of solutions be found? [*John Hoyt, Indiana University of Pennsylvania.*]

Solution: a. If n is even, then choose the permutation $(0,n-1,2,n-3,4,n-5,6,\dots,1)$. It is easy to show that the residues modulo n of the partial sums form the permutation $(0,n-1,1,n-2,2,n-3,\dots,n/2)$. Now suppose n is odd. Clearly, any acceptable permutation must have 0 as its first component, else two consecutive partial sums would be the same. But the last component of the sequence of partial sums reduced modulo n will also be 0, since $\Sigma_{i=1}^{n-1}i=n(n-1)/2$, which is divisible by n if n is odd. Thus it is possible to find such permutations if and only if n is even or $n=1$.

JORDAN I. LEVY, student
University of Delaware

Editor's comment. No complete solutions to part b were received. E. T. H. Wang has found with the aid of the computer that if $f(n)$ denotes the number of admissible solutions, then $f(1)=f(2)=1$, $f(4)=2$, $f(6)=4$, $f(8)=24$, and $f(10)=288$. John Hoyt comments that part b was suggested by a music composition student who wanted to write sequences of twelve different tones such that the successive intervals (measured in half tones) are different.

Part a was also solved by John Hoyt, Eli L. Isaacson, Edward T. H. Wang (Canada), and G. Wedderburn.

Reflections **January 1977**

1003. Let P and Q be two distinct points in the interior of a circular disk with neither point at the center. With the boundary of the disk acting as a mirror, a ray of light from point P determines, by the successive reflections from the boundary, a polygonal path in the disk. This path is dependent on the initial direction of the ray of light. Given a positive integer k, show that there is such a path with the kth reflection of the ray intersecting Q.

*With k, P and Q given, can the number of such distinct paths be determined? [*Richard Crandall, Boston, Massachusetts, and Peter Ørno, The Ohio State University.*]

Solution: Suppose the circular disk is the set of points A in the Cartesian plane with $|A| \leqslant 1$. Without loss of generality, we may assume that $|P| \geqslant |Q|$ and P lies on the positive x-axis. Consider first what happens to a ray of light which emanates from P straight upward (making angle $\pi/2$ with the positive x-axis). After any number of reflections, its closest approach to the center of the circle is distance $|P|$, and the center of the circle is to the left for someone traveling on the ray of light. Thus the directed line segment from the kth reflection to the $(k+1)$th reflection has Q on its left (unless $|P|=|Q|$ and it happens to pass through Q). The same argument shows that when a ray emanates from P straight downward (angle $-\pi/2$), the directed line segment following the kth reflection has Q on its right. As the angle of emanation of the ray varies from $-\pi/2$ to $\pi/2$, the line segment varies continuously, from a position with Q on the right to a position with Q on the left, so for some angle between, the kth reflection passes through Q.

Comments: Determining the number of solutions seems to be hard. If the above argument is put in analytic form, it turns out that a ray emanating from the point $(r,0)$ making angle α with the positive real axis passes through the point (x,y) after k reflections if and only if $f(\alpha)=0$, where f is defined by:

$$f(\alpha)=x\cos\delta+y\sin\delta-r\sin\alpha,$$
$$\text{where } \delta=\alpha+k(\pi-2\sin^{-1}(r\sin\alpha))-\frac{\pi}{2}.$$

This is an analytic function of α, with period 2π, so there are only finitely many solutions α with $-\pi<\alpha\leqslant\pi$. This formula is of appropriate size for a programmable calculator. Some calculated results are: From (.5,0) to (.1,.1), $1\leqslant k\leqslant 5$, 2 solutions in each case; From (9,0) to (−.8,.4), $k=1$, 4 solutions, $k=2$, 8 solutions, $k=3$, 10 solutions.

PETER ØRNO
The Ohio State University

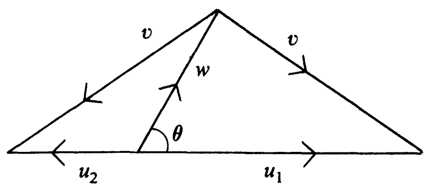
First part also solved by Bert Gunter and Philip Straffin. Partial solution by Michael Goldberg.

Fastest and Slowest Trip **January 1977**

1004. A river flows with a constant speed w. A motorboat cruises with a constant speed v with respect to the river, where $v>w$. If the path travelled by the boat is a square of side L with respect to the ground, the time of the traverse will vary with the orientation of the square. Determine the maximum and minimum time for the traverse. [*M. S. Klamkin, University of Alberta.*]

Solution: To travel along a side of the square making an angle $\theta\leqslant\pi/2$ with the direction of current, the motorboat must be set in an appropriate direction, as shown in the diagram below, and the resultant speed is

$$u_1=\sqrt{v^2-w^2\sin^2\theta}+w\cos\theta.$$



The same diagram also shows that the resultant speed along the opposite side is

$$u_2=\sqrt{v^2-w^2\sin^2\theta}-w\cos\theta.$$

Replacing θ by $(\pi/2)-\theta$, we obtain the resultant speeds u_3 and u_4 along the other two sides. The time T of traverse is $L(u_1^{-1}+u_2^{-1}+u_3^{-1}+u_4^{-1})$. Thus, we find that

$$T=\frac{2L}{v^2-w^2}\left[2v^2-w^2+\sqrt{(2v^2-w^2)^2-(w^2\cos 2\theta)^2}\,\right]^{1/2}.$$

We see that the minimum occurs when $\theta=0$ and $T_{\min}=2L(v+\sqrt{v^2-w^2})/(v^2-w^2)$. The maximum occurs when $\theta=\pi/4$ and $T_{\max}=2L(4v^2-2w^2)^{1/2}/(v^2-w^2)$.

PAUL Y. H. YIU
University of Hong Kong

Also solved by John Brunn, Albert C. Claus, Jordi Dou (Spain), Howard Eves, Marguerite F. Gerstell, Michael Goldberg, George C. Harrison, Vaclav Konecny, William Myers, Scott Smith, J. M. Stark, Philip Straffin, H. Ziehms (Germany), and the proposer. There was one unsigned solution from the University of Texas.

A Popular Characterization **January 1977**

1005. Suppose f and g are differentiable functions for $x>0$ and $f'(x)=-g(x)/x$ and $g'(x)=-f(x)/x$. Characterize all such f and g. [*Brian Hogan, Highline Community College, Midway, Washington.*]

Solution I: We have

$$[x(f(x)+g(x))]'=xf'(x)+xg'(x)+f(x)+g(x)=0.$$

Thus $f(x)+g(x)=2A/x$ for some constant A. Similarly, we have

$$[(f(x)-g(x))/x]'=[xf'(x)-xg'(x)-f(x)+g(x)]/x^2=0.$$

Thus $f(x)-g(x)=2Bx$ for some constant B. This yields

$$f(x)=\frac{A}{x}+Bx \quad \text{and} \quad g(x)=\frac{A}{x}-Bx$$

for arbitrary constants A and B.

KENNETH KLINGER
University of Arizona

Solution II: Differentiating f' and using the given relations to eliminate g and g', we find $x^2f''(x)+xf'(x)-f(x)=0$. This is an Euler differential equation. If we try $f(x)=x^a$, we see that $a(a-1)+a-1=0$ has $a=\pm 1$ as solutions. Thus the general solution is $f(x)=cx+d/x$ and then $g(x)=-cx+d/x$.

PHILIP STRAFFIN
Beloit College

Also solved by John Atkins, Augusta College Problem Solving Group, J. D. Baum, Stanley J. Becker, J. C. Binz (Switzerland), M. T. Bird, Michael Brozinsky, Dale Burton & Johnny Henderson, Bruce R. Caine, Paul Chauveheid (Belgium), Winnie Cho, Charles Chouteau, Albert C. Claus, Terry L. Cleveland, Clayton W. Dodge, Ragnar Dybvik (Norway), Michael W. Ecker, Michael J. Evans, David Farnsworth, Robert S. Fisk, Daniel S. Freed, Donald C. Fuller, Ralph Garfield, Marguerite F. Gerstell, Michael Goldberg, M. R. Gopal, Richard A. Groeneveld, Lee O. Hagglund, George C. Harrison, G. A. Heuer, J. D. Hiscocks (Canada), Eli L. Isaacson, Richard Johnsonbaugh, Lester W. Jones, Jr., Ole Jørsboe (Denmark), Steven Kahan, Steve Kahn, Geoffrey A. Kandall, Mark Kleiman, William R. Klinger, Neil Levy, Denis Lichtman, Henry S. Lieberman, Peter A. Lindstrom, Frederick M. Liss, Timothy A. Loughlin, Roger F. McCouch, Arthur S. McDade, James C. McKim, R. B. McNeill, B. Margolis (Bolivia), William Myers, Larry O. Olson, P. J. Pedler, Gary D. Peterson, Doug Pfendler, Adam Riese, C. L. Sabharwal, Henry Schultz, David Scott, H. T. Sedinger, Joseph Silverman, Arthur Solomon, Blair Spearman, Neville Spenser, Joseph J. Spila, J. M. Stark, Richard K. Stark, James P. Stoessel & M. Z. Williams, K. H. S. Subrahmanyam (India), Pambuccian Victor (Romania), G. Walther (Germany), Edward T. H. Wang (Canada), J. S. Wasileski, Samuel Weinberger, Paul Y. H. Yiu (Hong Kong), Ken Yocum, Robert L. Young, Michael Zielinski, and the proposer. There was one unsigned solution.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q655. Let S denote the left hand side of the inequality. Then S can be rewritten as

$$\frac{a(d+a)+c(b+c)}{(b+c)(d+a)}+\frac{b(a+b)+d(c+d)}{(c+d)(a+b)}.$$

Since $(x+y)^2\geqslant 4xy$,

$$S\geqslant 4\left\{\frac{a(d+a)+c(b+c)}{(a+b+c+d)^2}+\frac{b(a+b)+d(c+d)}{(a+b+c+d)^2}\right\}=2\frac{(a+b+c+d)^2+(a-c)^2+(b-d)^2}{(a+b+c+d)^2}\geqslant 2.$$

Equality holds if and only if $a=c$ and $b=d$.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Pierre Malraison, Editor
Control Data Corp.

Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Some reviews of books are adapted from the Telegraphic Reviews in the American Mathematical Monthly.

Fitzgerald, Anne and MacLane, Saunders (Eds), Pure and Applied Mathematics in the People's Republic of China: A Trip Report of the American Pure and Applied Mathematics Delegation, National Academy of Sciences, 1977; ix + 116 pp, $8.25 (P).

A detailed report of the 1976 visit of nine mathematicians to universities, factories, and a mathematics institute in mainland China. A shorter account by Victor Klee appeared in *Amer. Math. Monthly* 84 (1977) 509-517.

Kolata, Gina Bari, *The end of elegance: the computer invades mathematics*, The Sciences 17:3 (May-June 1977) 6-9.

What will mathematics of the future be like? Three recent uses of computers may foreshadow changes: Haken and Appel's computer-aided proof of the Four Color Theorem, a result in genealogy by Ulam and Marr inspired by computer-generated statistical results, and Rabin's computerized error-prone test for primality.

An uncrackable code? Time, 3 July 1978, 55-56.

Popular account of new developments in cryptology based on factorizations of large primes.

Four mathematicians get Fields Medals, Science News 114:8 (August 19, 1978) 119.

Brief announcement of awards at the August 1978 International Congress of Mathematicians in Helsinki to Daniel Quillen, Charles Fefferman, Pierre Deligne and G.A. Margulis.

Pippenger, Nicholas, *Complexity theory*, Scientific American 238:6 (June 1978) 114-124, 160.

Complexity theory in the specific sense meant here refers to determining the minimal number of components needed to perform a given task in a device such as a telephone exchange or computer, which have large numbers of simple components. The author recounts recent progress toward more efficient telephone networks, both in new combinatorial designs and in estimating lower bounds for the number of required components.

Greitzer, Samuel L. (Ed), International Mathematical Olympiads 1959-1977. New Math. Lib., V. 27. MAA, 1978; xi + 204 pp, $6.50 (P).

Contains all of the problems (with solutions) for the first 19 Olympiads: a splendid collection of elementary problems.

Kolata, Gina Bari, *"Geodesy: dealing with an enormous computer task,"* Science 200 (28 April 1978) 421-422, 466.

The U.S. National Geodetic Survey is engaged in a 10-year task of readjusting the North American Datum, a network of reference points whose longitudes, latitudes, and altitudes are known very accurately. A key step will involve solving 2.5 million non-linear equations in 400,000 unknowns, the largest system ever attempted. *Taylor series* will be used to linearize the equations; their number will be reduced to 400,000 by *least squares* adjustment, which guarantees a unique solution to the resulting 400,000 x 400,000 *linear system*. The solution will then be effected piecewise through the use of *block matrices*.

Diaconis, Persi, *Statistical problems in ESP research,* Science 201 (14 July 1978) 131-136.

Critical evaluation of ESP experiments by a statistician who also happens to be a professional magician. New statistical methods have been developed to deal with complex feedback and freedom of response common in ESP research; even so, Diaconis reports, controls are frequently so loose that no valid statistical analysis is possible.

Dieudonné, Jean, *Present trends in pure mathematics,* Advances in Mathematics 27 (March 1978) 235-255.

"A summary of a summary of a summary"--of the author's book *Panorama des mathêmatiques pures: Le choix bourbachique* (Gauthier-Villars, 1977), itself a guide among the 500-odd "exposés" of the Séminaire Bourbaki. A small number of general observations is followed by progress reports of achievements in individual branches since 1940. Now, who will attempt the like for applied mathematics?

Pluperfect square, Scientific American 238:6 (June 1978) 86, 88.

Recounts determination of smallest n(=112) for which an n x n square can be dissected into smaller squares of all-different integral sides.

Beniger, James R. and Robyn, Dorothy L., *Quantitative graphics in statistics: a brief history,* American Statistician 32:1 (February 1978) 1-11.

Did you ever wonder who invented semi-log paper (Jevous)? or bar charts (Playfair, in 1786)? or ogives (Fourier, 1821)? or time-line charts (Priestley)? or when co-ordinate paper was first printed (1794)? These and other innovations have been as influential as developments in notation, and this article outlines their history.

Findler, Nicholas V., *Computer poker,* Scientific American 239:1 (July 1978) 144-151, 162.

Draw poker--"the American game"--is a suitable vehicle of investigation by computer science into the nature of human judgment. When is an opponent likely to be bluffing? What is a judicious use of bluffing? Experiments with human players have led to various machine strategies (both static and learning), and comparisons among them are presented.

Papert, Seymour A., *The mathematical unconscious,* in Wechsler, Judith (Ed), On Aesthetics in Science, MIT Pr, 1978; pp. 104-119.

"Popular views of mathematics...exaggerate its logical face and devalue all connections with everything else in human experience. By so doing, they fail to recognize the resonances between mathematics and the total human being which are responsible for mathematical pleasure and beauty." This essay proceeds from Poincaré's theory of mathematical thinking to reflect on how the lost aesthetic dimension should be reincorporated in mathematical education.

Robinson, A.L., *Computer films: adding an extra dimension to research*, Science 200 (26 May 1978) 749-752.

Applications of computer graphics to chemistry and biology, also discusses state-of-the-art technology for films.

Gardner, M., *Mathematical games*, Scientific American 239:2 (August 1978) 18-25.

A Moebius strip is really a twisted prism. This article discusses generalizations to other cross sections and number of twists, as well as questions of slicing. Connections with number theory (sides vs. twists depends on g.c.d.'s) and impossible figures (is the box for the three pronged blivet really a twisted prism?).

Brams, Steven J., Comparison Voting, Test Edition, American Political Science Association, 1978; 80 pp, (P).

This preliminary edition of a political science module compares different voting systems and concludes that "approval voting" offers important advantages. Approval voting, a compromise between current U.S. practice and systems of ranking, allows a voter to cast a vote for each of one or more candidates; the candidate with the most votes is elected. Advantages include simplicity, easy implementation on existing voting machines, and elimination of third-party "spoilers" (under a mild condition, approval voting leads to election of the "Condorcet winner"--and eliminates any benefit of insincere or strategic voting). This module is an attractive unit for a course in finite or liberal arts mathematics, since it endeavors to teach in gentle fashion and in an applied context the logic of mathematical reasoning. Further mathematical details can be found in: Brams, Steven J. and Fishburn, Peter C., *Approval voting*, American Political Science Review (1978).

Fujimura, Kobon, The Tokyo Puzzles. Transl: Fumie Adachi. Scribner's, 1978; 184 pp, $8.95.

98 classic puzzles told by Japan's most popular writer of puzzle books. Many themes are well known in the West (false coins, moving matchsticks, logical liars) but cast here with novel angles; others are not known widely outside Japan. Try this: cut a cube of cheese into two pieces each with a cross section shaped like a regular hexagon.

Gardner, Martin, *The charms of catastrophe*, The New York Review (15 June 1978) 30-33.

Extensive review of four books on catastrophe theory (by Thom, Zeeman, Woodcock and Davis, Poston and Stewart), including a layman's introduction to the theory and a personal evaluation of the great CT debate.

Sussman, Hector J. and Zahler, Raphael S., *Catastrophe theory: mathematics misused*, The Sciences 17:6 (October 1977) 20-23.

Popular account of the author's misgivings, which were also expounded in *Nature* and *Science* and reviewed here earlier.

Machol, Robert E., *et al.*, Management Science in Sports, North-Holland/TIMS, 1976; xiii + 164 pp, $14 (P).

This companion volume to Ladany and Machol's *Optimal Strategies in Sports* appeared as a special issue of *Management Science;* it differs in including only scholarly articles. Four deal with American football, and one each with baseball, weight-lifting, tennis, hockey, cross-country scoring and the long jump. Others treat the sports draft, ticket pricing, scheduling, and ranking of teams.

NEWS & LETTERS

ALLENDOERFER, FORD, POLYA AWARDS

Authors of seven expository papers published in 1977 issues of journals of the Mathematical Association of America received awards at the 1978 August meeting of the Association at Brown University. The 1977 awards, each in the amount of $100, are:

Carl B. Allendoerfer Awards:

David A. Smith, Human Population Growth: Stability or Explosion? *Mathematics Magazine* 50 (1977) 186-197.

Branko Grunbaum and Geoffrey C. Shephard, Tilings by Regular Polygons, *Mathematics Magazine* 50 (1977) 227-247.

Lester R. Ford Awards:

T.F. Banchoff and L.H. Kauffman, Immersions and mod-2 Quadratic Forms, *Amer. Math. Monthly* 84 (1977) 168-185.

Ralph P. Boas, Partial Sums of Infinite Series, and How They Grow, *Amer. Math. Monthly* 84 (1977) 237-258.

Neil J.A. Sloane, Error-Correcting Codes and Invariant Theory: New Applications of a Nineteenth-Century Technique, *Amer. Math. Monthly* 84 (1977) 82-107.

George Pōlya Awards:

Allen H. Holmes, John LeDuc and Walter Sanders, Statistical Inference for the General Education Student--It Can Be Done, *TYCMJ* 8 (1977) 223-230.

Frieda Zames, Surface Area and the Cylinder Area Paradox, *TYCMJ* 8 (1977) 207-211.

1978 FIELDS MEDALS

The four 1978 Fields Medals, awarded on August 15, 1978 at the Eighteenth International Congress of Mathematicians in Helsinki, Finland, went to Daniel Quillen of the Massachusetts Institute of Technology, Charles Fefferman of Princeton University, Pierre Deligne of the Institut des Hautes Etudes Scientifiques in France and G.A. Margulis of the Soviet Union. Regarded as the equivalent for mathematics of the Nobel Prizes, the Fields Medals are awarded every four years to mathematicians under the age of 40.

Quillen was cited in his award as being the prime architect of algebraic K theory, a new tool that has successfully employed geometric and topological tools to solve major problems in algebra.

Fefferman, winner in 1976 of the National Science Foundation's Alan T. Waterman award, was cited for several innovations that have revived the study of multidimensional complex analysis by finding correct generalizations of classical (low dimensional) results.

Deligne, from Belgium, is best known for his 1974 solution of the three Weil conjectures concerning generalizations of the Riemann Hypothesis to finite fields. His work has done much to unify algebraic geometry and algebraic number theory.

Margulis received his Fields Medal for innovative analysis of the structure of Lie Groups. These groups are now playing a key role in current work in high energy particle physics.

TRUTH ABOUT FALSE COINS

In a recent letter (this *Magazine*, November 1977, p. 277) Steven Conrad suggested that Maurice Kraitchik may deserve credit for the false coin problem, which appears at the end (pp. 324-325) of the 1953 Dover edition of Kraitchik's book *Mathematical Recreations*. Bennet Manvel (this *Magazine*, March 1977, pp. 90-92) had attributed the problem to E.D. Schell (*Amer. Math. Monthly*, 42 (1945) 397). The issue is whether the problem that appears in the 1953 Dover edition of *Mathematical Recreations* also appeared in the original 1942 Norton edition.

There is strong circumstantial evidence that it did not. In the Dover edition, there are anomalies of type face, section numbering, and page layout. Spurred by Conrad's comment, I undertook to put the matter to rest. At the Library of Congress, I obtained for inspection a copy of the 1942 edition. The false-coin subsection was missing. Anecdotally, Michael Goldberg provided further confirmation: in personal conversation with Schell, he verified that in 1945 the problem was original with Schell.

Benjamin L. Schwartz
216 Apple Blossom Ct.
Vienna
Virginia 22180

1978 INTERNATIONAL MATHEMATICAL OLYMPIAD

The twentieth International Mathematical Olympiad, won by Romania, was held in Bucharest, Romania, on July 5 and 6, 1978. The United States team, coached by Samuel Greitzer and Murray Klamkin, finished second among the eighteen teams that entered the competition; one member of the U.S. team, Mark Kleiman of New York City, achieved the only perfect score. The exam consists of the following six problems:

1. m and n are natural numbers with $n > m \geq 1$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value. *(Cuba)*

2. P is a given point inside a given sphere and A, B, C are any three points on the sphere such that PA, PB and PC are mutually perpendicular. Let Q be the vertex diagonally opposite to P in the parallelepiped determined by PA, PB and PC. Find the locus of Q. *(USA)*

3. The set of all positive integers is the union of two disjoint subsets

$$\{f(1), f(2), \ldots f(n), \ldots\}$$
$$\{g(1), g(2), \ldots g(n), \ldots\},$$

where

$$f(1) < f(2) < \ldots < f(n) < \ldots,$$
$$g(1) < g(2) < \ldots < g(n) < \ldots,$$

and

$$g(n) = f(f(n)) + 1 \quad \text{for all} \quad n \geq 1.$$

Determine $f(240)$. *(Gt. Britain)*

4. In triangle ABC, $AB = AC$. A circle is tangent internally to the circumcircle of Triangle ABC and also to sides AB, AC at P, Q respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC. *(USA)*

5. Let $\{a_k\}$ $(k = 1,2,3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geq \sum_{k=1}^{n} \frac{1}{n}.$$ *(France)*

6. An international society has its members from six different countries. The list of members contains 1978 names, numbered 1, 2, ... , 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country. *(Netherlands)*

ANECDOTES WANTED

We wish to publish a collection of anecdotes about well-known mathematicians. If you are interested in contributing, please write to

Peter Borwein
Department of Mathematics
University of British Columbia
Vancouver, B.C., Canada V6T 1W5

or

Maria Klawe
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For each anecdote please include your source and your assessment of its truth (as a probability between 0 and 1).

METRIC IMITATIONS

The main result (first theorem) of Ira Rosenholtz' note "Imitating the Euclidean Metric" (this *Magazine*, March 1978, pp. 125-126) is essentially a restatement of the following theorem due to Herbert Busemann: A Hausdorff space possesses a finitely compact metrization iff it is locally compact and has a countable base (*Trans. Amer. Math. Soc.*, 56 (1944) 205). By a "finitely compact" metric space he meant one in which every bounded sequence has a convergent subsequence, and he used the fact that in such a metric space every open ball has compact closure. Indeed, it is easily verified that the two properties are equivalent. Thus Rosenholtz' theorem is equivalent to Busemann's.

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Editor's Note: Nearly every issue of *Mathematics Magazine* contains letters similar to this one, pointing out earlier work that contained the essence of a recently published *Magazine* paper. We print these letters in the interest of historical honesty as an informational service to readers. The fact that certain results appeared before in the mathematical literature is by now quite common: who, indeed, can vouchsafe that his theorem has never appeared anywhere in the nearly 2000 journals that publish mathematical research?

We would like to stress, however, that originality of results is not one of the main editorial objectives of *Mathematics Magazine*. While we are not indisposed to original theorems, we are far more interested in innovative presentations, scholarly surveys, and careful exposition. New ways of looking at old ideas can be just as instructive as old ways of looking at new ideas, especially if the presentation makes interesting material readily accessible to a large audience.

QUICK--THE ANSWER

My reaction to the note "Tossing Coins Until All Are Heads" (this *Magazine*, May 1978, p. 184) was to accept as a challenge the problem the author left unsolved: express $E(Y)$ in a computationally accurate manner. The following approach works quite well.

Given n and a number $0 \leq q < 1$ let $a_k = (1-q^k)^n$, for $1 \leq k$. The sequence $\{a_k\}$ is increasing, so the sum $b_k = \Sigma_{r=1}^{k-1} a_r$ is quite accurately found, relative to the computational accuracy of the a_r. Now the value $E(Y)$ is easily seen to be the limit of $S_k = ka_k - b_k$. This solves the problem.

I found the following values using $q = .5$ ("convergence" occurred at $k = 41$):

n	S_{41}	b_{41}(approx)
50	6.990977903	34
75	7.571163305	33
100	7.983801534	33
150	8.566369982	32
200	8.980204217	32
300	9.563969331	31
1000	11.299252682	30
10000	14.62053125	26
100000	17.94239101	23

I expect that these give accurate values for $E(Y)$ except for possibly the last digit. Thanks for proposing the problem of computational accuracy.

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WAR WITHOUT END

In Gary Sherman's "A child's game with permutations" (this *Magazine*, January 1978, pp. 67-68), the following question was proposed: Does the game of war ever have to end? The answer is, "No," regardless of "local" stalemate rules and of the number of ranks used, a rank being 4 of a kind. To see this first consider two players A and B each having 26 cards. Let A's cards from top to bottom (face down) be

(K Q J 10 9 8 7 6 5 4 3 2 K
1 Q J 10 9 8 7 6 5 4 3 2 1)

where K = king, Q = queen, J = jack, and 1 = ace. Let B's cards be

(Q K 10 J 8 9 6 7 4 5 2 3 1
K J Q 9 10 7 8 5 6 3 4 1 2).

Here we shall consider ace as low. It is easily seen that by properly placing the cards at the bottom of the winning player's pile after each play, each player can at the end of 26 plays have the same cards (without regards to suits) in the same order as they were in the beginning of the game. Thus, the game could go on forever regardless of stalemate rules since no stalemates occur.

If on the other hand the kings are discarded from the original deck of cards, a similar result follows with A's cards

(Q J 10 9 8 7 6 5 4 3 2 1
Q J 10 9 8 7 6 5 4 3 2 1)

and B's cards

(J Q 9 10 7 8 5 6 3 4 1 2
J Q 9 10 7 8 5 6 3 4 1 2).

As a final example one may remove the 2's and 6's from the original deck of cards and consider A's cards

(K Q J 10 9 8 7 5 4 3 K
1 Q J 10 9 8 7 5 4 3 1)

and B's cards

(Q K 10 J 8 9 5 7 3 4 1
K J Q 9 10 7 8 4 5 1 3).

Similarly, any number of ranks and any ranks may be omitted from the original deck being used.

It is interesting to notice further that if A has a tendency to play his card down faster than B and if B then plays his card by overlapping it on A's, then each time the cards are picked up and placed at the bottom of the winning player's pile (in the usual manner), they are in the desired order to keep the game going on indefinitely.

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ERRATA

The equation involving angle C of the type 10 pentagon in Table II of my paper "Tiling the Plane with Congruent Pentagons" (this *Magazine*, January 1978, pp. 29-44) should be: $C = \pi - D/2$.

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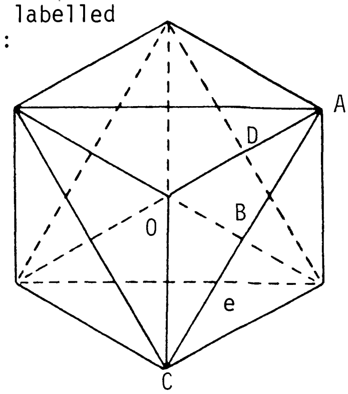
On p. 56 of my note "An Infinite Class of Deltahedra" (this *Magazine*, January 1978, pp. 55-56), the descriptive "ditetrahedra" in line 9 was inexplicably replaced with a somewhat redundant "deltahedra," and the identifying letters in Figure 3, referred to in the two paragraphs bracketing the Figure, were omitted. Here is the correctly labelled Figure:



Such are the vicissitudes of publishing.

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Volume 51, Number 4, September 1978